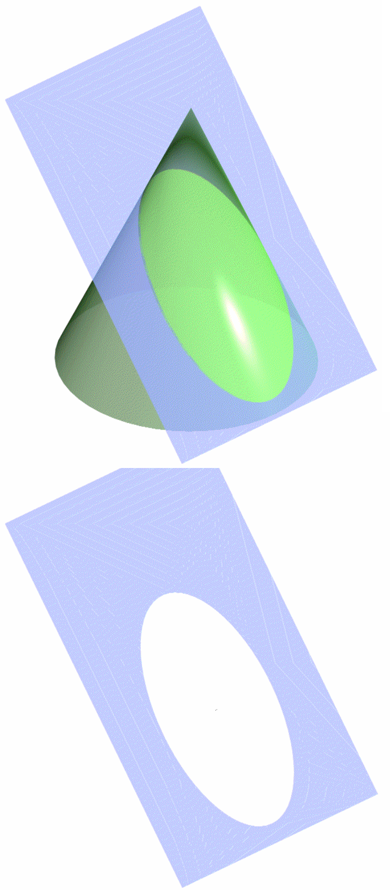
Descriptive Geometry 2

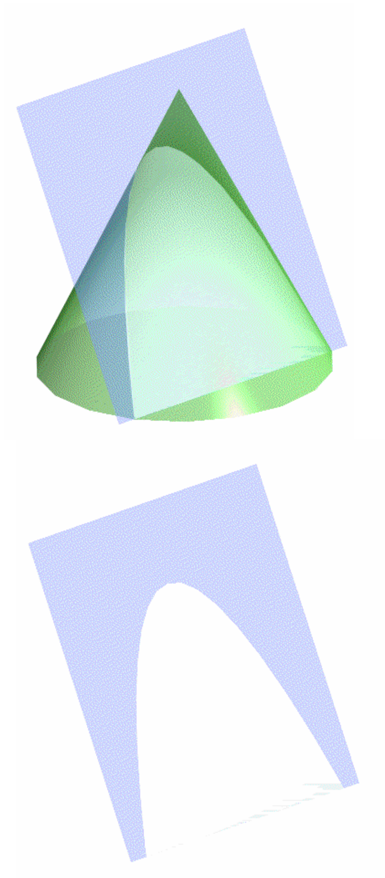
By Pál Ledneczki Ph.D.

Conic Sections

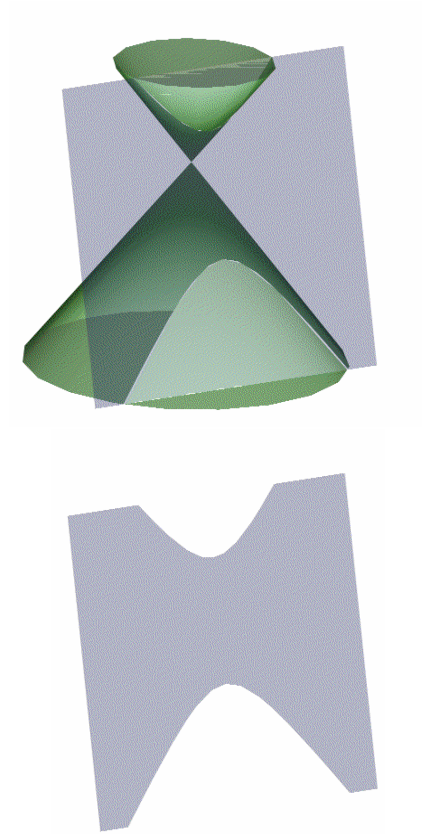
Ellipse



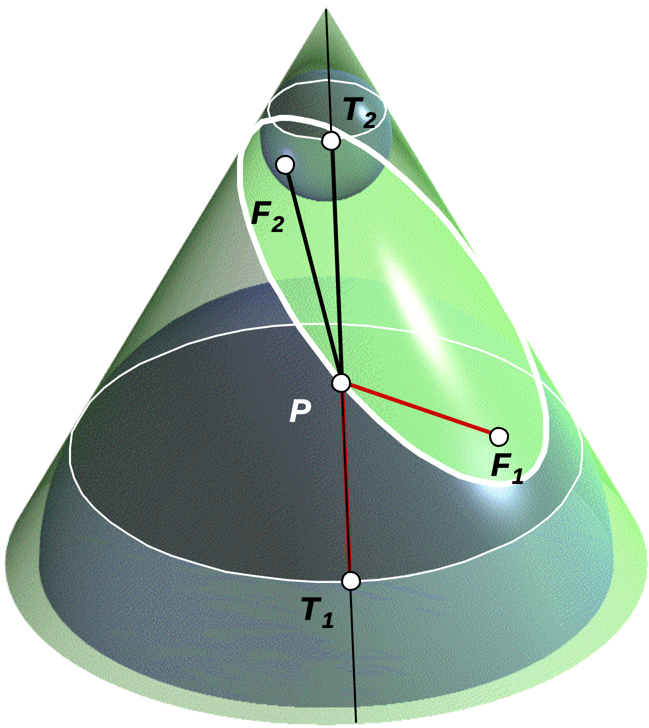
Parabola



Hyperbola



Intersection of Cone and Plane: Ellipse



The intersection of a cone of revolution and a plane is an *ellipse* if the plane (not passing through the vertex of the cone) intersects all generators.

Dandelin spheres: spheres in a cone, tangent to the cone (along a circle) and also tangent to the plane of intersection.

Foci of ellipse: F_1 and F_2, points of contact of plane of intersection and the Dadelin spheres.

P: piercing point of a generator, point of the curve of intersection.

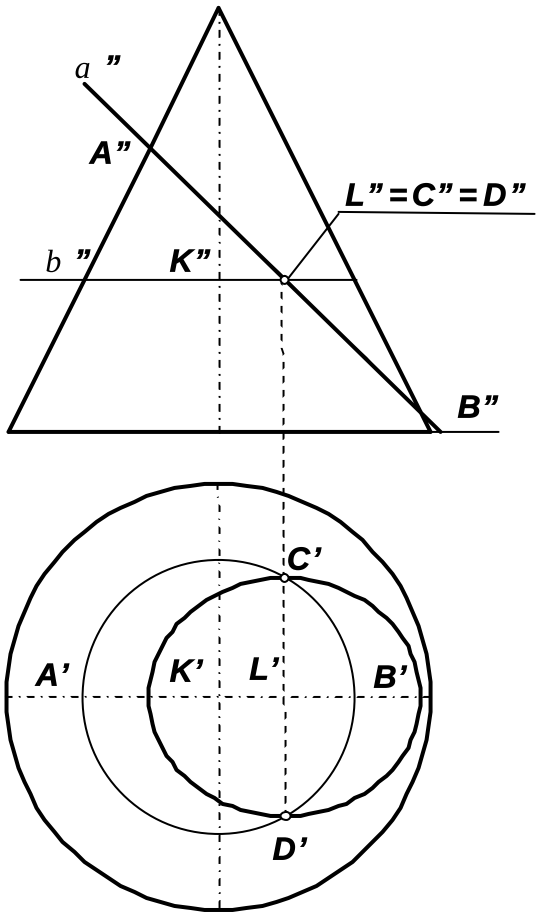
T_1 and T_2, points of contact of the generator and the Dandelin sphere.

$PF_1 = PT_1$, $PF_2 = PT_2$ (tangents to sphere from an external point).

$PF_1 + PF_2 = PT_1 + PT_2 = T_1T_2 =$ constant

http://www.clowder.net/hop/Dandelin/Dandelin.html

Construction of Minor Axis



Let the plane of intersection *a*” second projecting plane that intersects all generators.

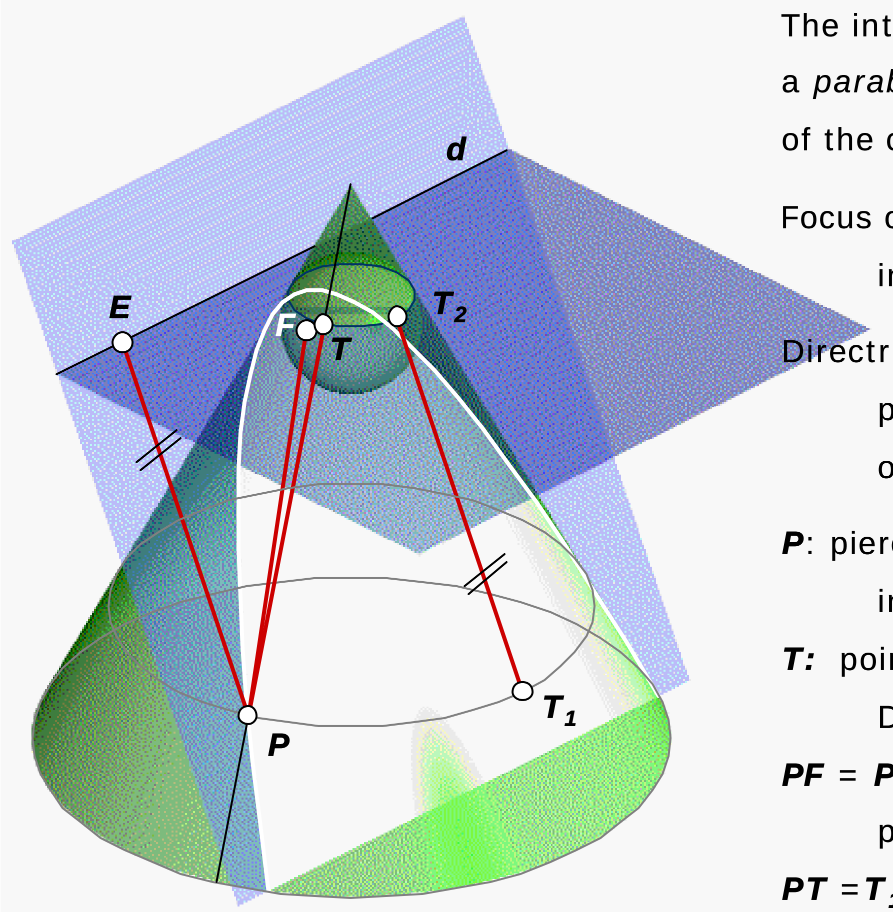
The endpoints of the major axis are ***A*** and ***B***, the piercing points of the leftmost and rightmost generators respectively.

The midpoint ***L*** of ***AB*** is the centre of ellipse.

Horizontal auxiliary plane *b*” passing through ***L*** intersects the cone in a circle with the centre of ***K.*** (***K*** is a point of the axis of the cone).

The endpoints of the minor axis ***C*** and ***D*** can be found as the points of intersection of the circle in *b* and the reference line passing through ***L”***.

Intersection of Cone and Plane: Parabola



http://mathworld.wolfram.com/DandelinSpheres.html

The intersection of a cone of revolution and a plane is a *parabola* if the plane (not passing through the vertex of the cone) is parallel to one generator.

Focus of parabola: ***F***, point of contact of the plane of intersection and the Dadelin sphere.

Directrix of parabola: ***d***, line of intersection of the plane of intersection and the plane of the circle on the Dandelin sphere.

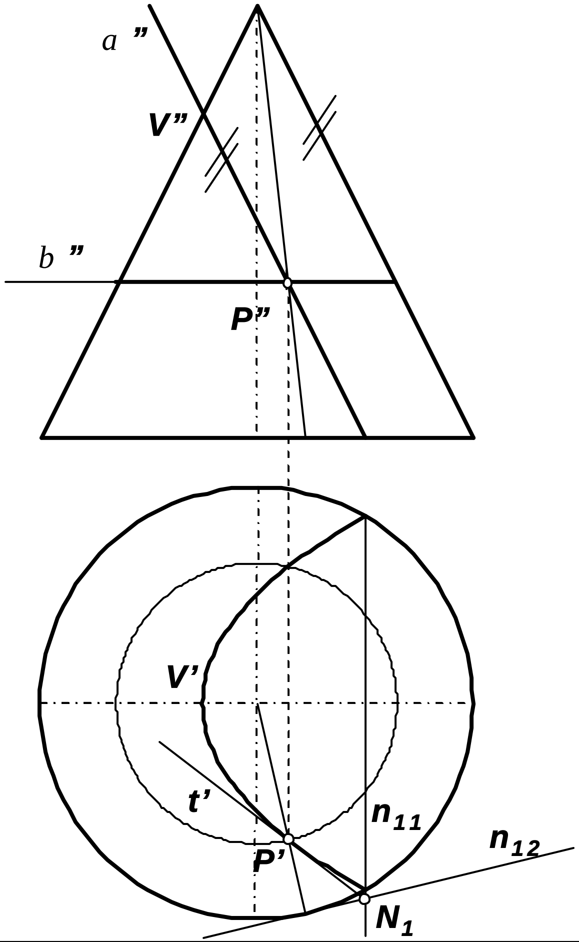
P: piercing point of a generator, point of the curve of intersection.

T: point of contact of the generator and the Dandeli sphere.

PF = ***PT*** (tangents to sphere from an external point).

$PT = T_1T_2 = PE$, $dist(P,F) = dist(P,d)$.

Construction of Point and Tangent



Let the plane of intersection *a”* second projecting plane be parallel to the rightmost generator.

The vertex of the parabola is ***V***.

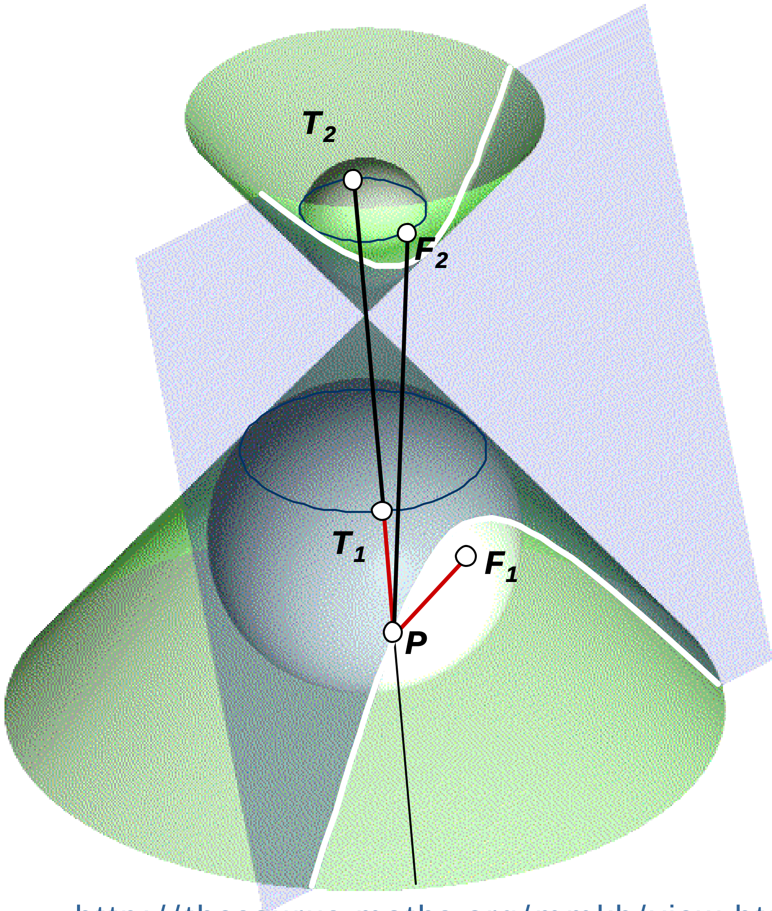
Horizontal auxiliary plane *b”* can be used to find ***P”***, the second image of a point of the parabola.

The tangent ***t*** at a point ***P*** is the line of intersection of the plane of intersection and the tangent plane of the surface at ***P***.

The first tracing point of the tangent N_1 is the point of intersection of the first tracing line of the plane of intersection and the firs tracing line of the tangent plane at ***P***, n_{11} and n_{12} respectively.

$$t = | N_1 P |$$

Intersection of Cone and Plane: Hyperbola



http://thesaurus.maths.org/mmkb/view.html

The intersection of a cone of revolution and a plane is a *hyperbola* if the plane (not passing through the vertex of the cone) is parallel to two generators.

Foci of hyperbola: F_1 and F_2, points of contact of plane of intersection and the Dadelin spheres.

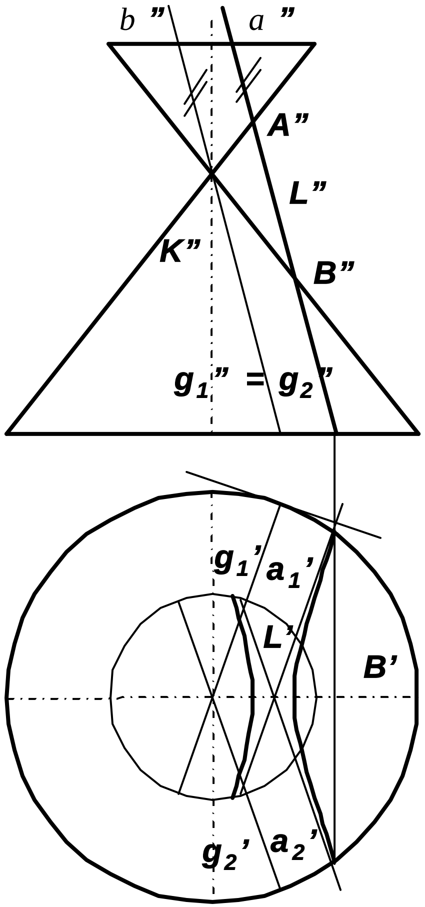
P: piercing point of a generator, point of the curve of intersection.

T_1 and T_2, points of contact of the generator and the Dandelin spheres.

$PF_1 = PT_1$, $PF_2 = PT_2$ (tangents to sphere from an external point).

$PF_2 - PF_1 = PT_2 - PT_1 = T_1T_2 =$ constant.

Construction of Asymptotes



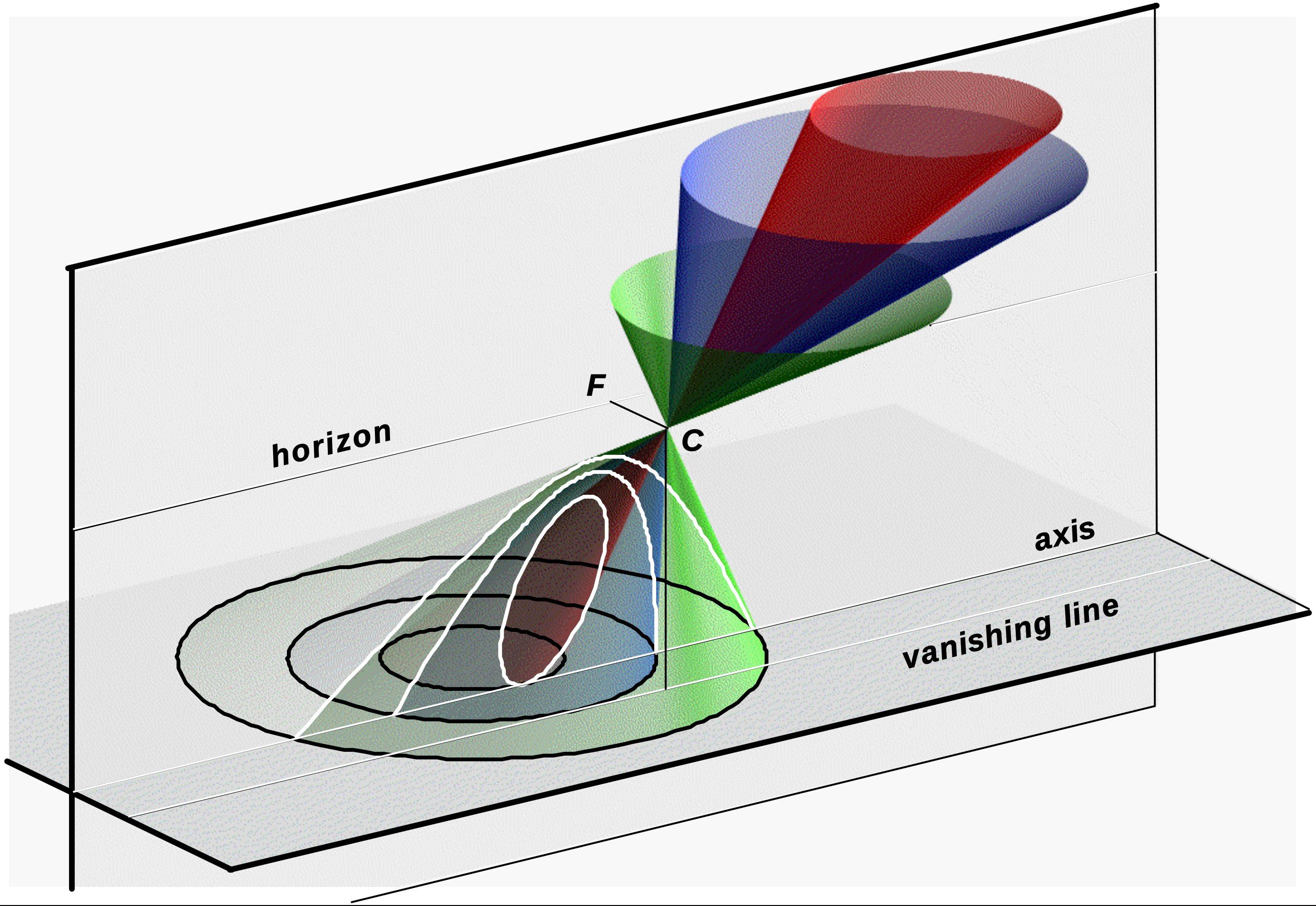
Let the plane of intersection *a*” second projecting plane parallel to two generators, that means, parallel to the second projecting plane *b* through the vertex of the cone, which intersects the cone in two generators g_1 and g_2.

The endpoints of the traverse (real) axis are ***A*** and ***B***, the piercing points of the two extreme generators.

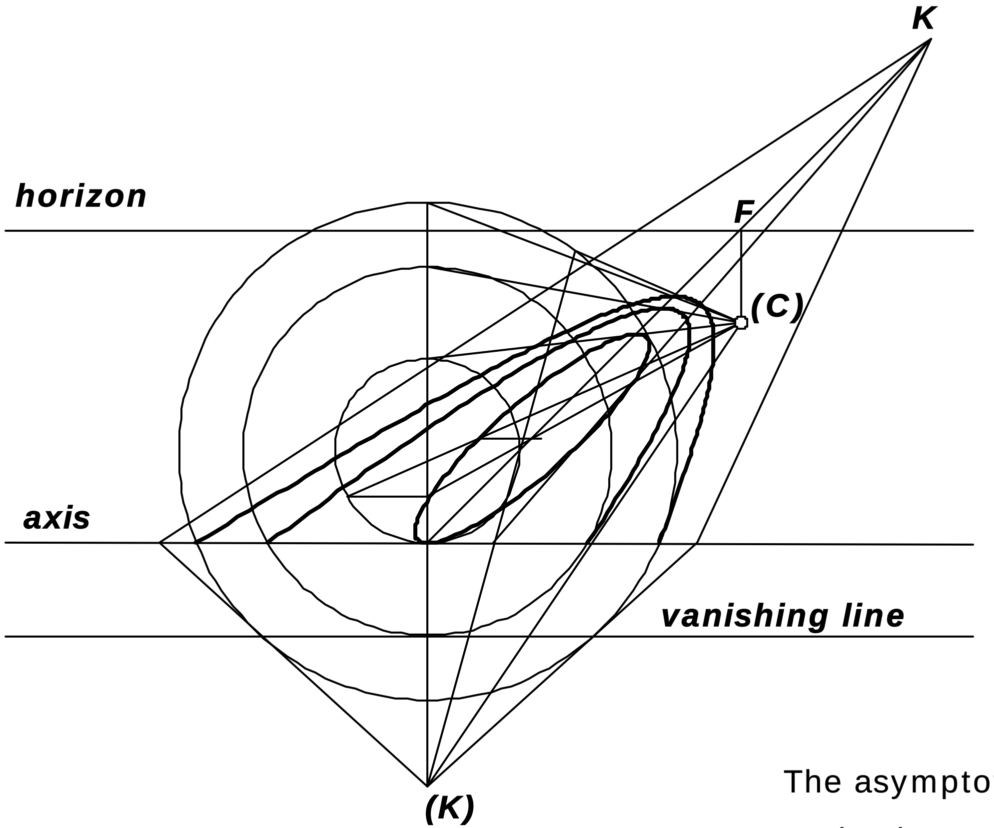
The midpoint ***L*** of ***AB*** is the centre of hyperbola.

The asymptotic lines a_1 and a_2 are the lines of intersections of the tangent planes along the generators g_1 and g_2 and the plane of intersection *a*.

Perspective Image of Circle



Construction of Perspective Image of Circle



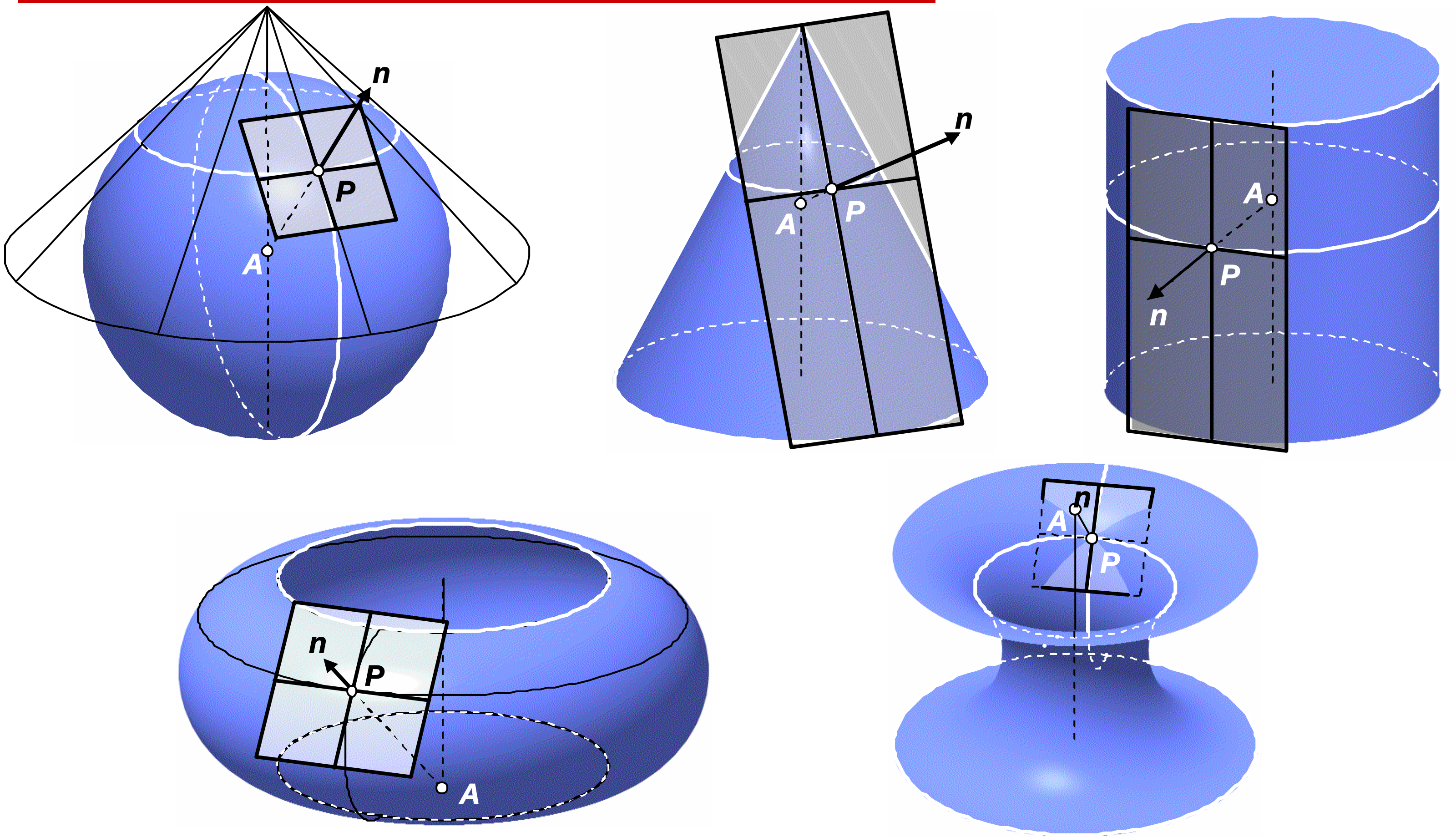
The distance of the horizon and the ***(C)*** is equal to the distance of the axis and the vanishing line.

The type of the perspective image of a circles depends on the number of common points with the vanishing line:

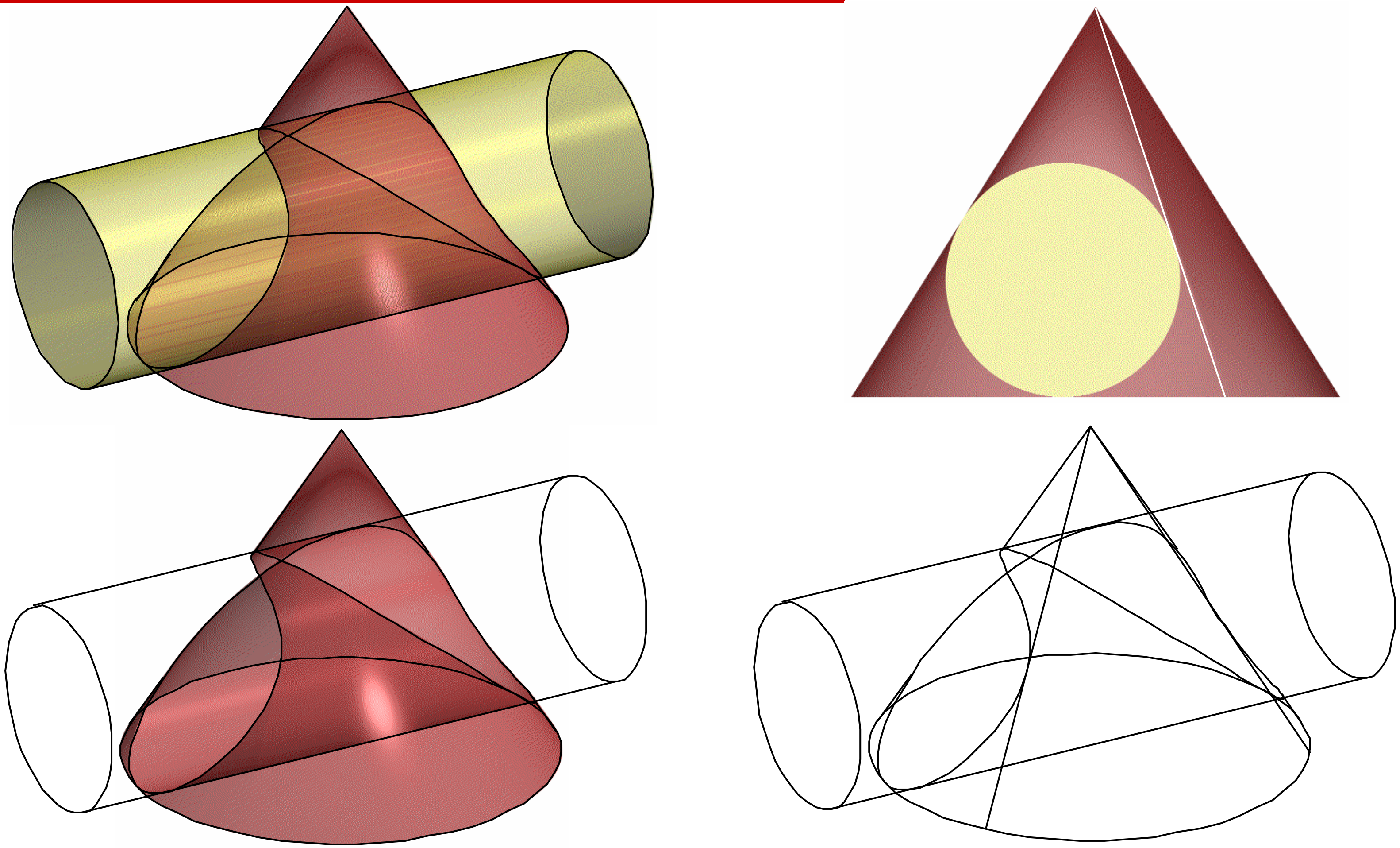
- no point in common; ellipse
- one point in common; parabola
- two points in common; hyperbola

The asymptotes of the hyperbola are the projections of the tangents at the vanishing points.

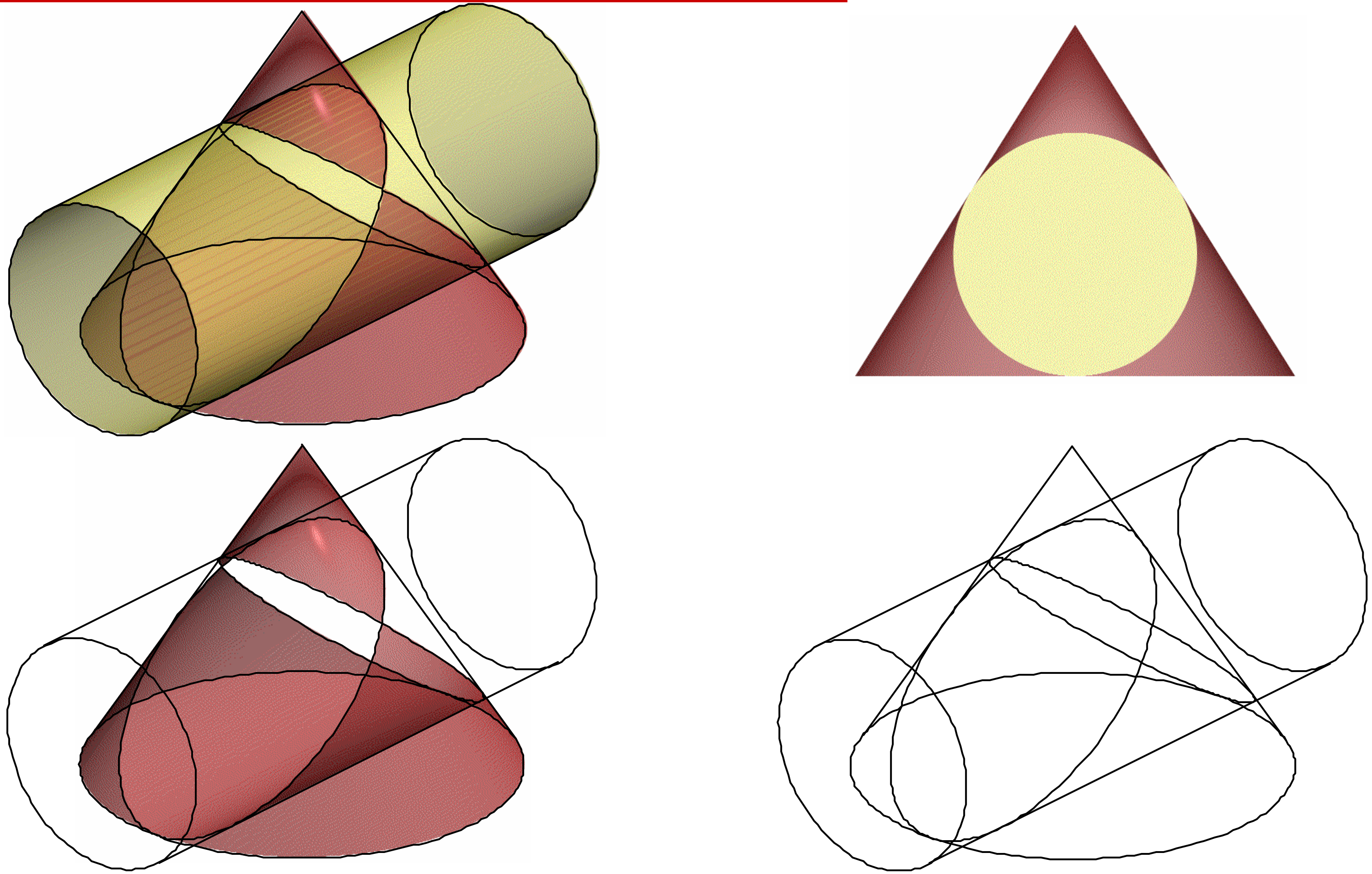
Tangent Planes, Surface Normals



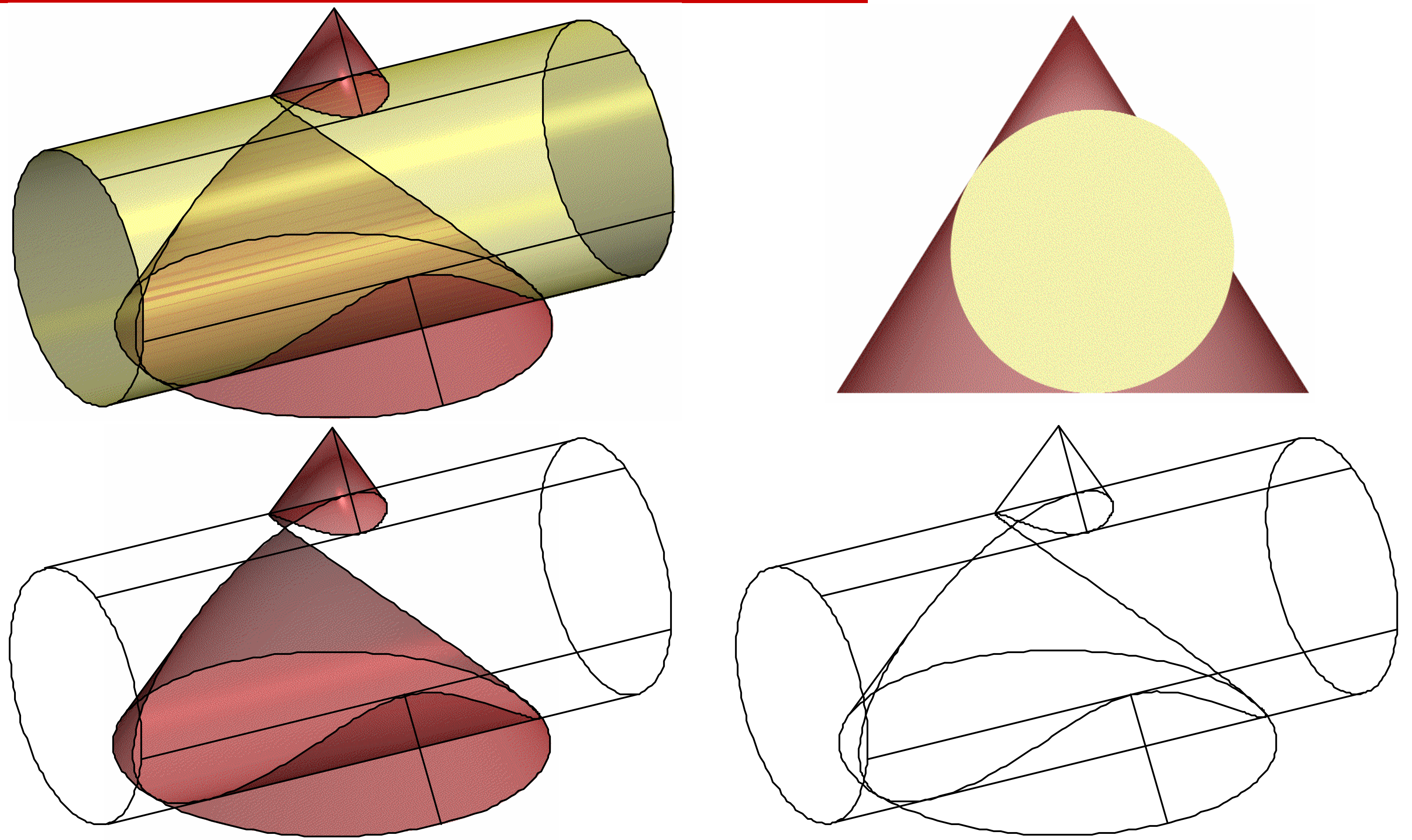
Intersection of Cone and Cylinder 1



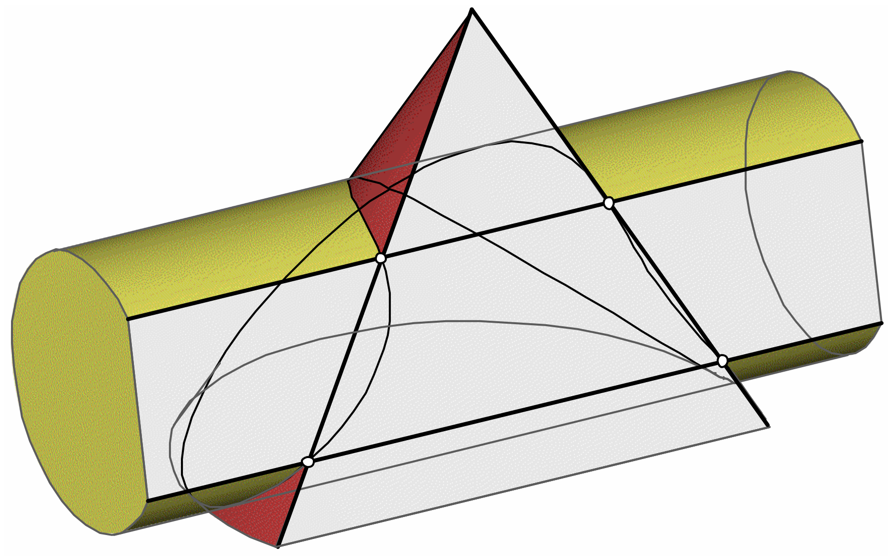
Intersection of Cone and Cylinder 2

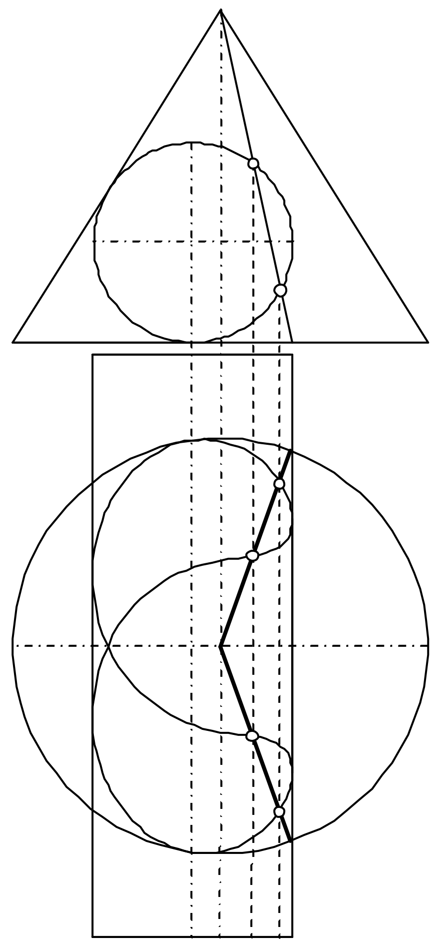


Intersection of Cone and Cylinder 3



Methods for Construction of Point 1

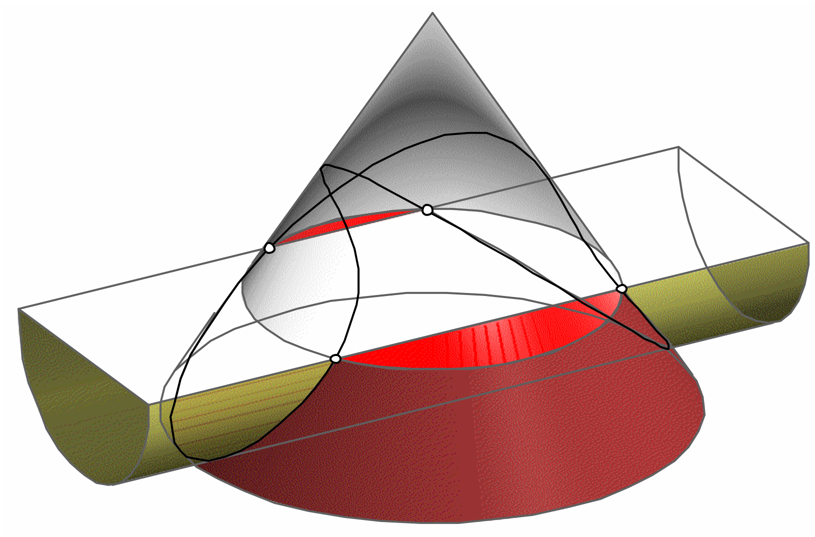


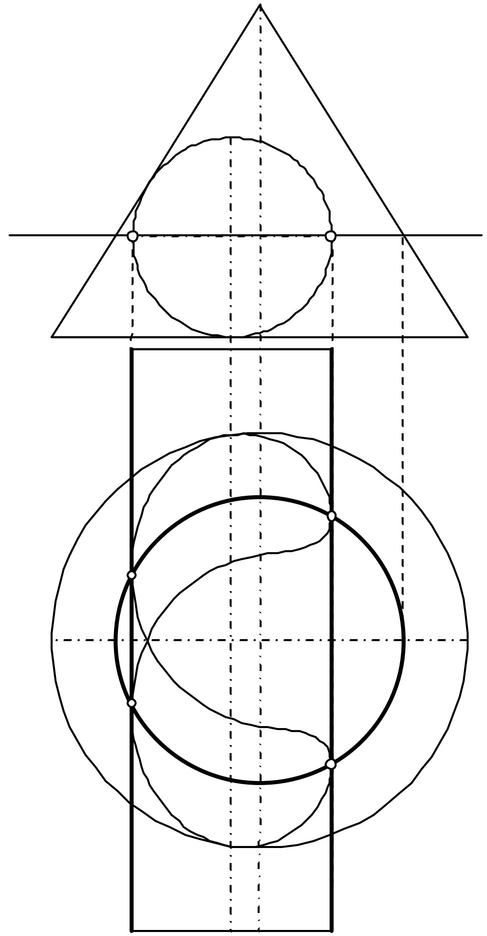


Auxiliary plane:

plane passing through the vertex of the cone
and parallel to the axis of cylinder

Methods for Construction of Point 2

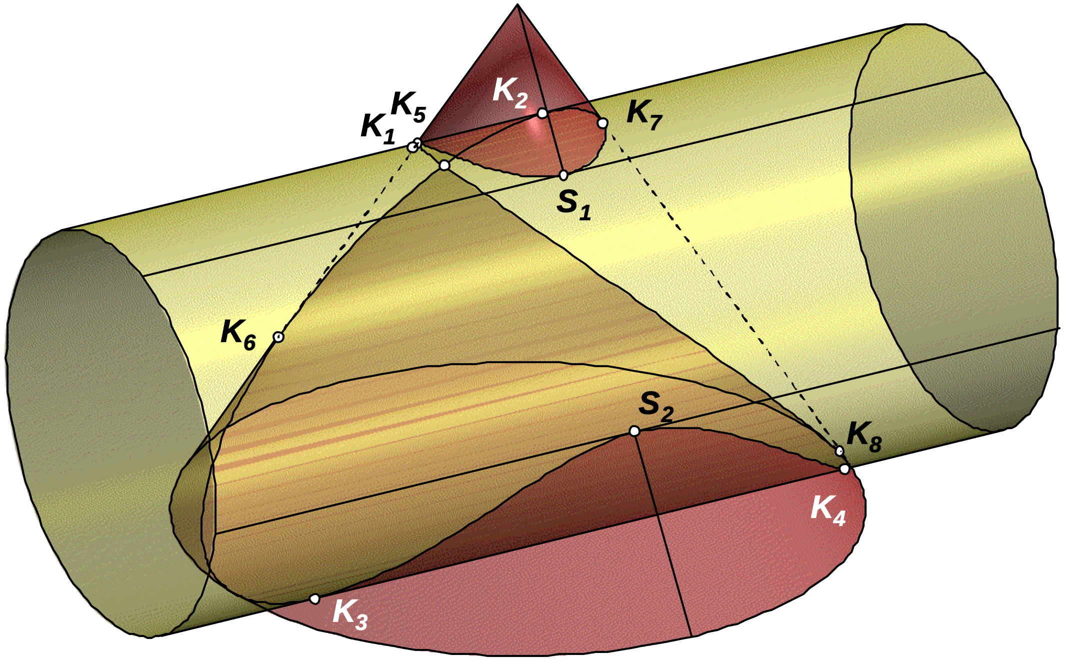


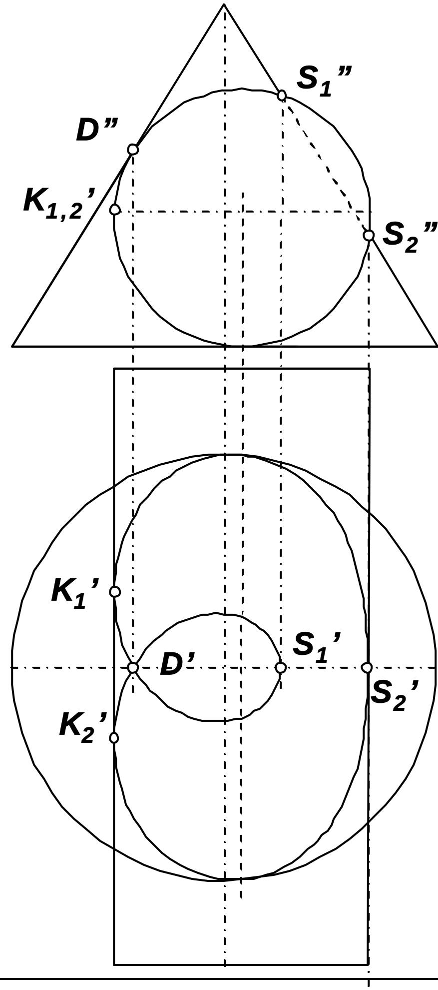


Auxiliary plane:

first principal plane

Principal Points





Double point: ***D***

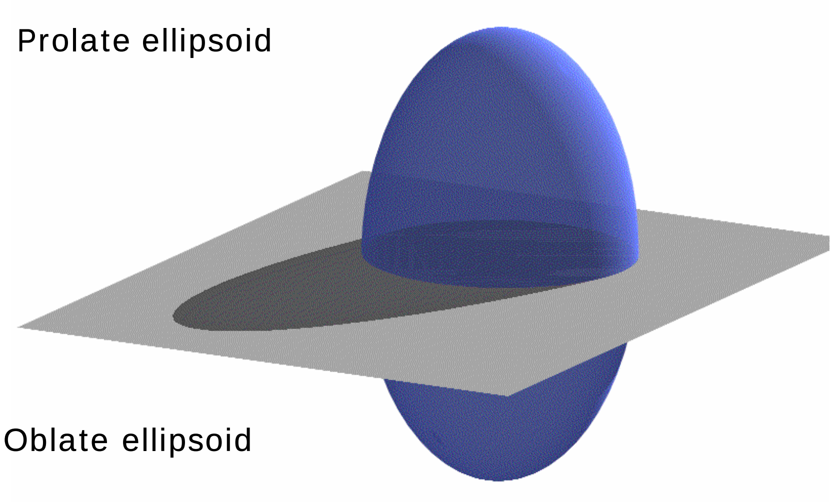
Points in the plane of symmetry: S_1, S_2

Points On the outline of cylinder: K_1, K_2, K_3, K_4

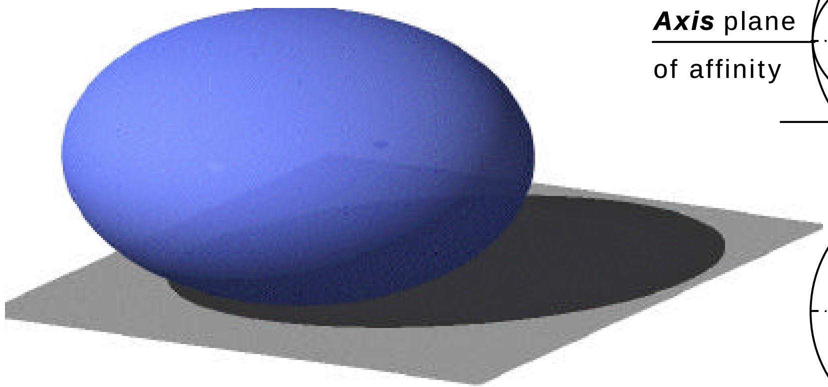
Points On the outline of cone: K_5, K_6, K_7, K_8

Surfaces of Revolution: Ellipsoid

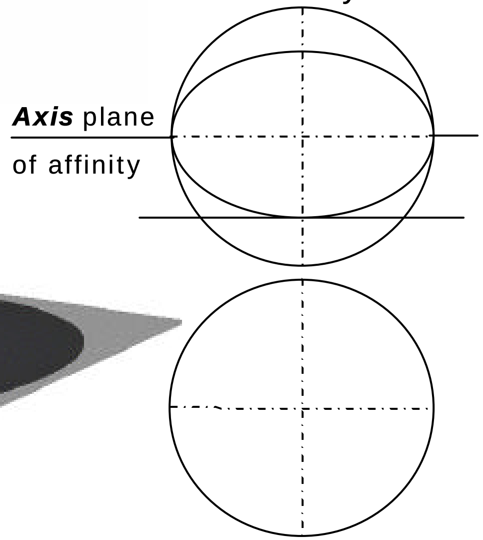
Prolate ellipsoid



Oblate ellipsoid

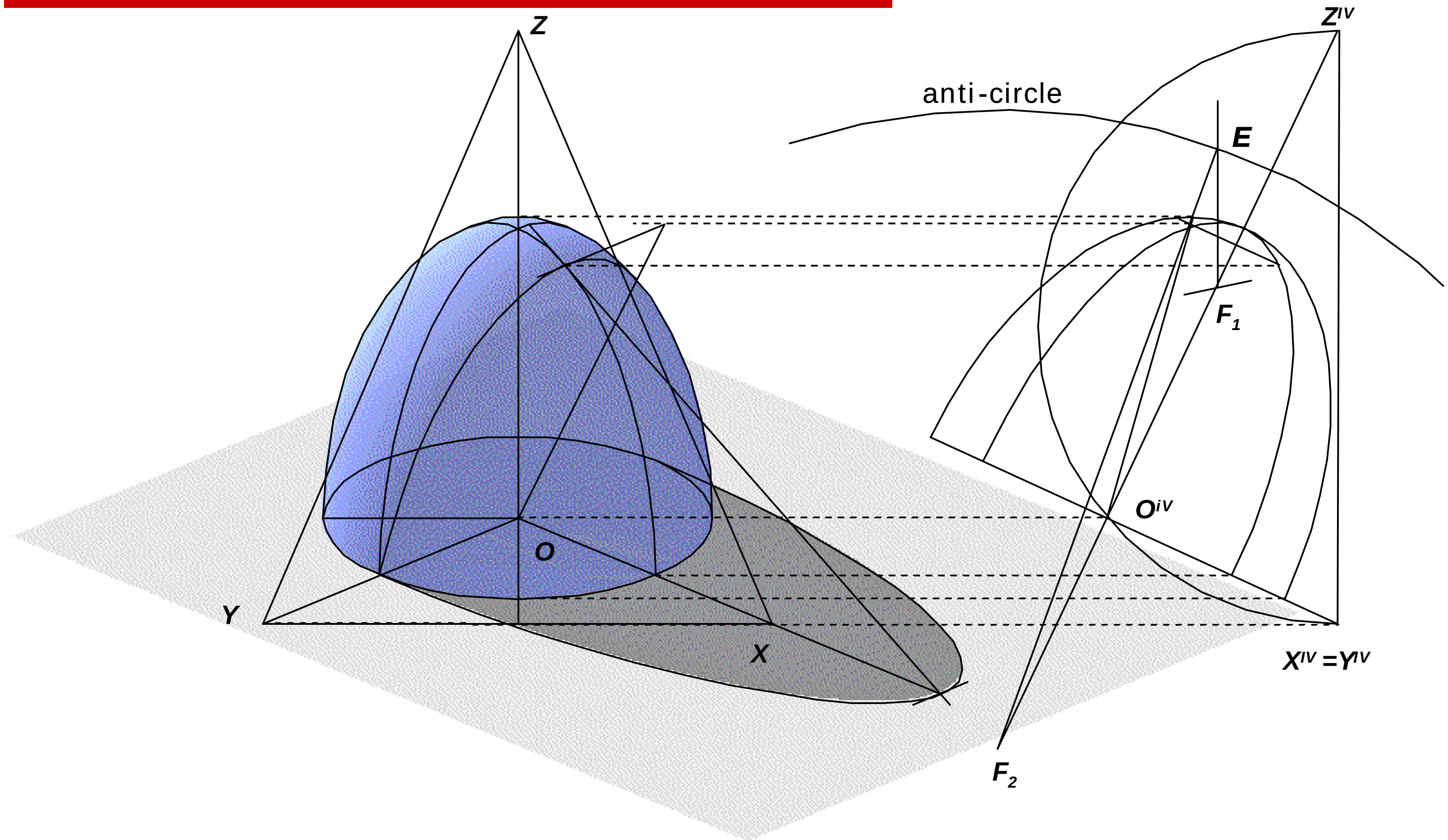


Affinity

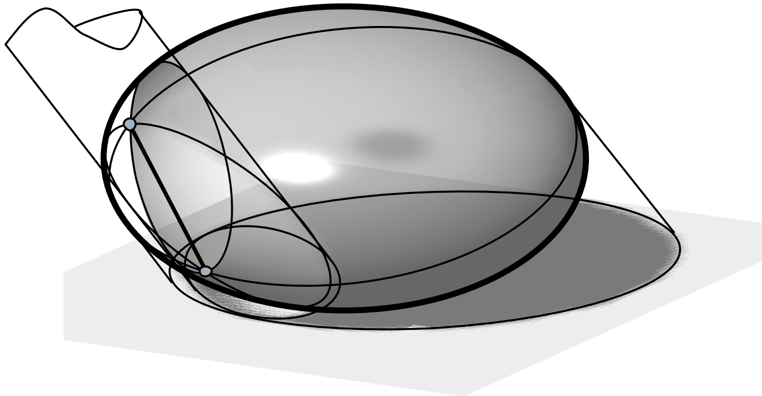


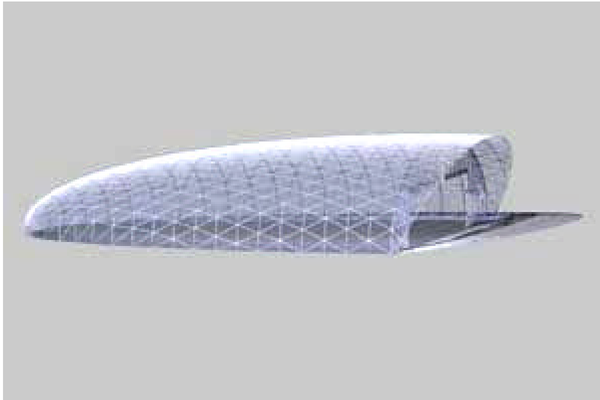
Capitol

Ellipsoid of Revolution in Orthogonal Axonometry

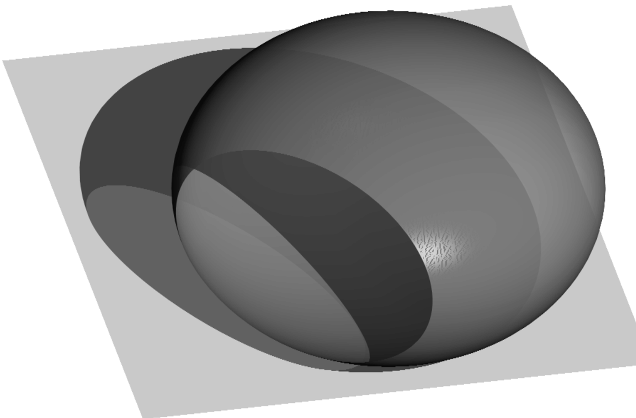


Intersection of Ellipsoid and Plane

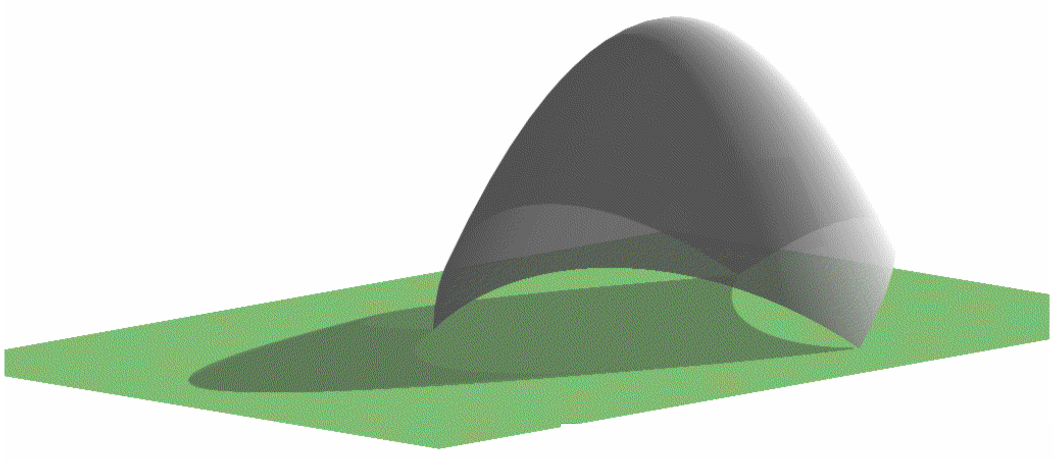


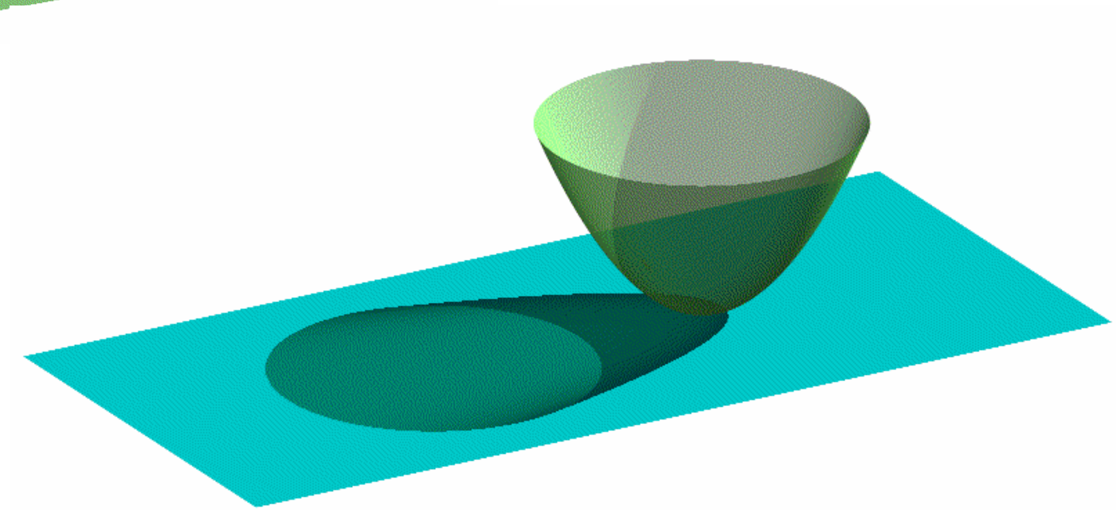


http://www.burgstaller-arch.at/

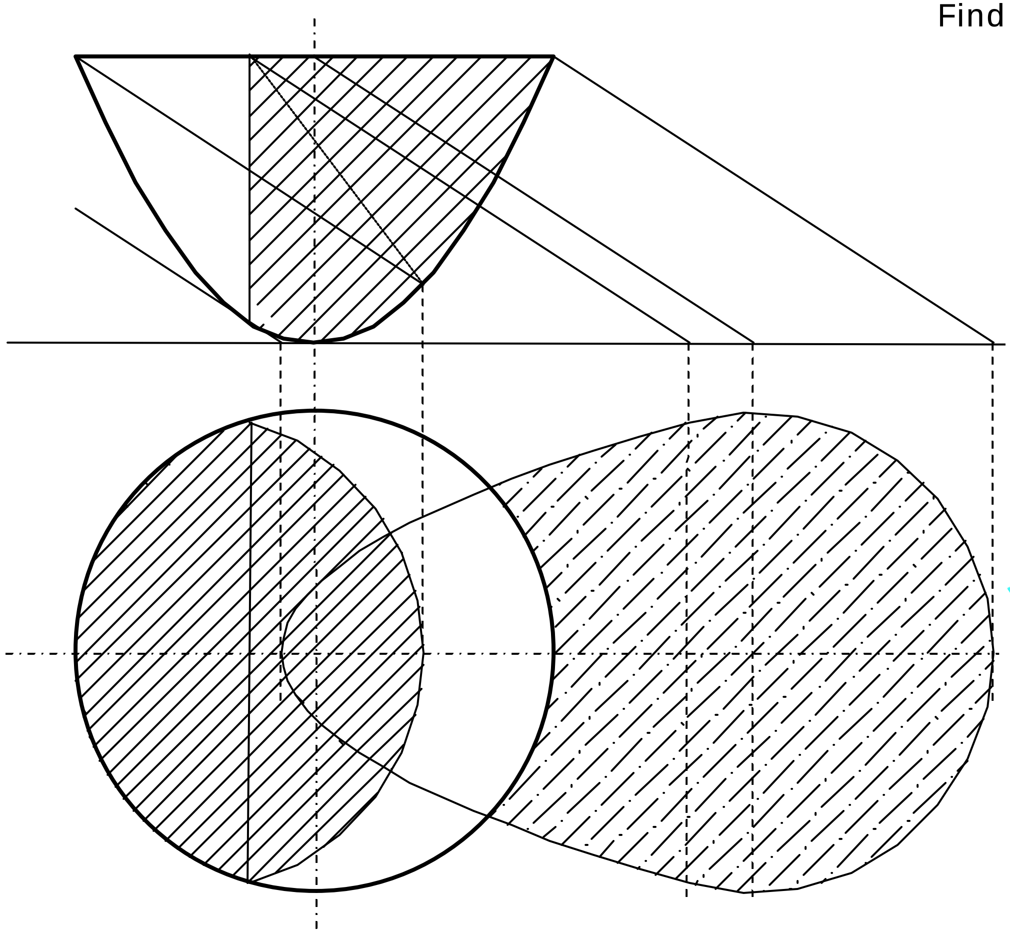


Surfaces of Revolution: Paraboloid



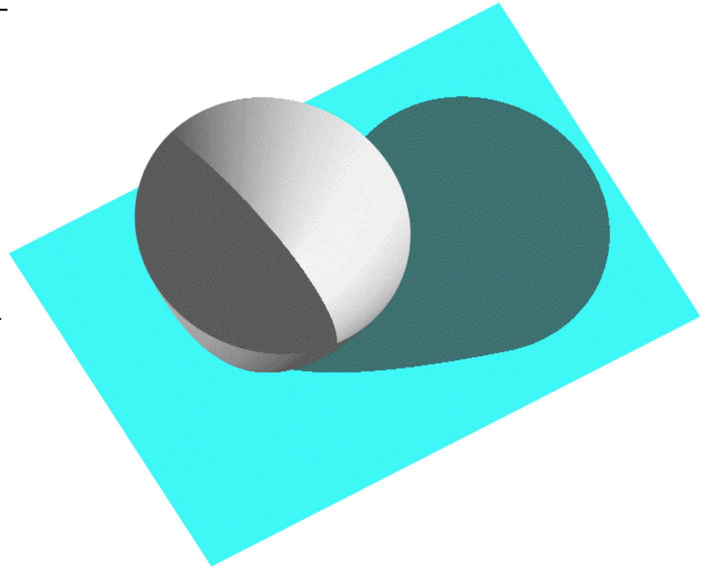


Paraboloid; Shadows

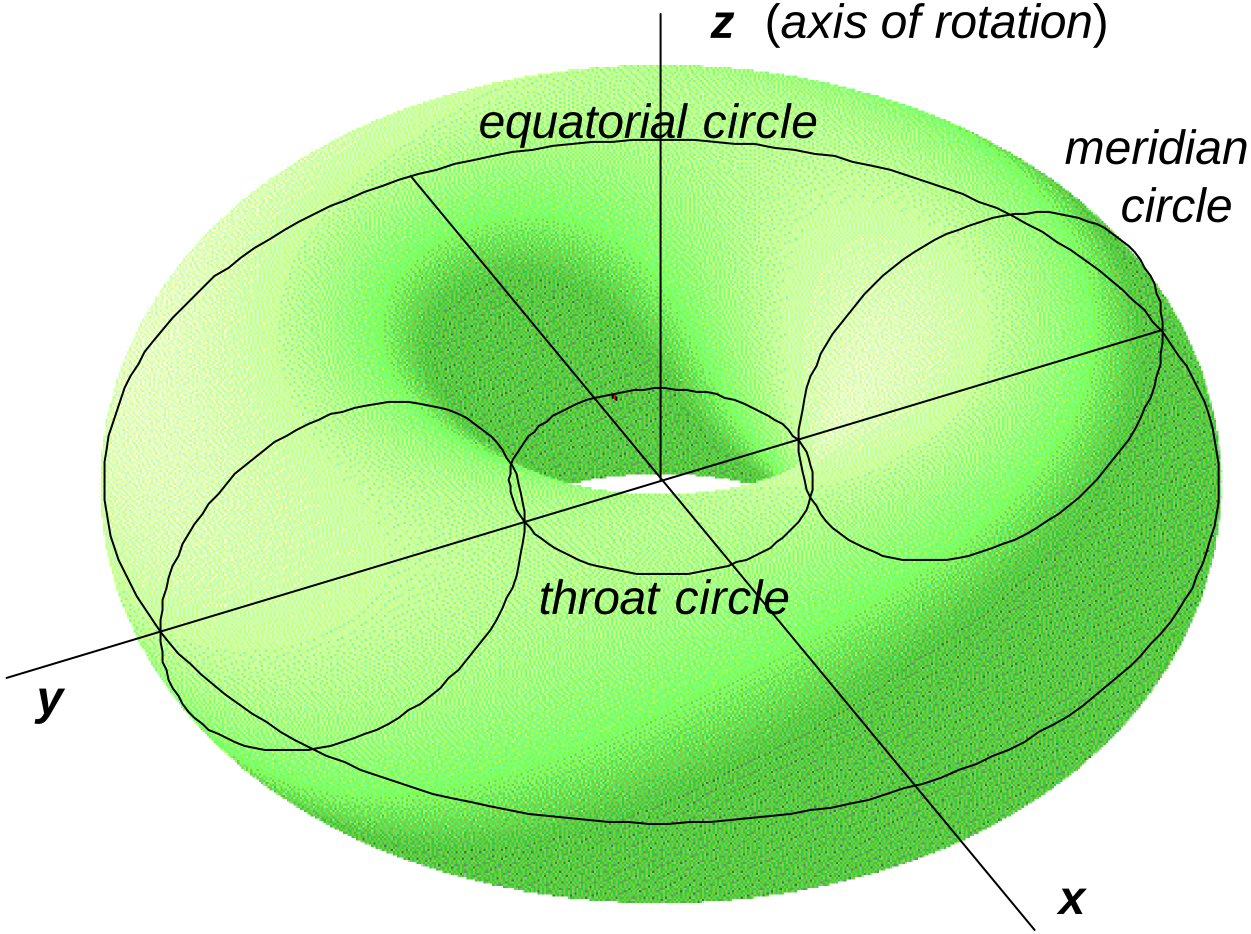


Find

- the focus of parabola
- tangent parallel to f”
- self-shadow
- cast shadow
- projected shadow inside

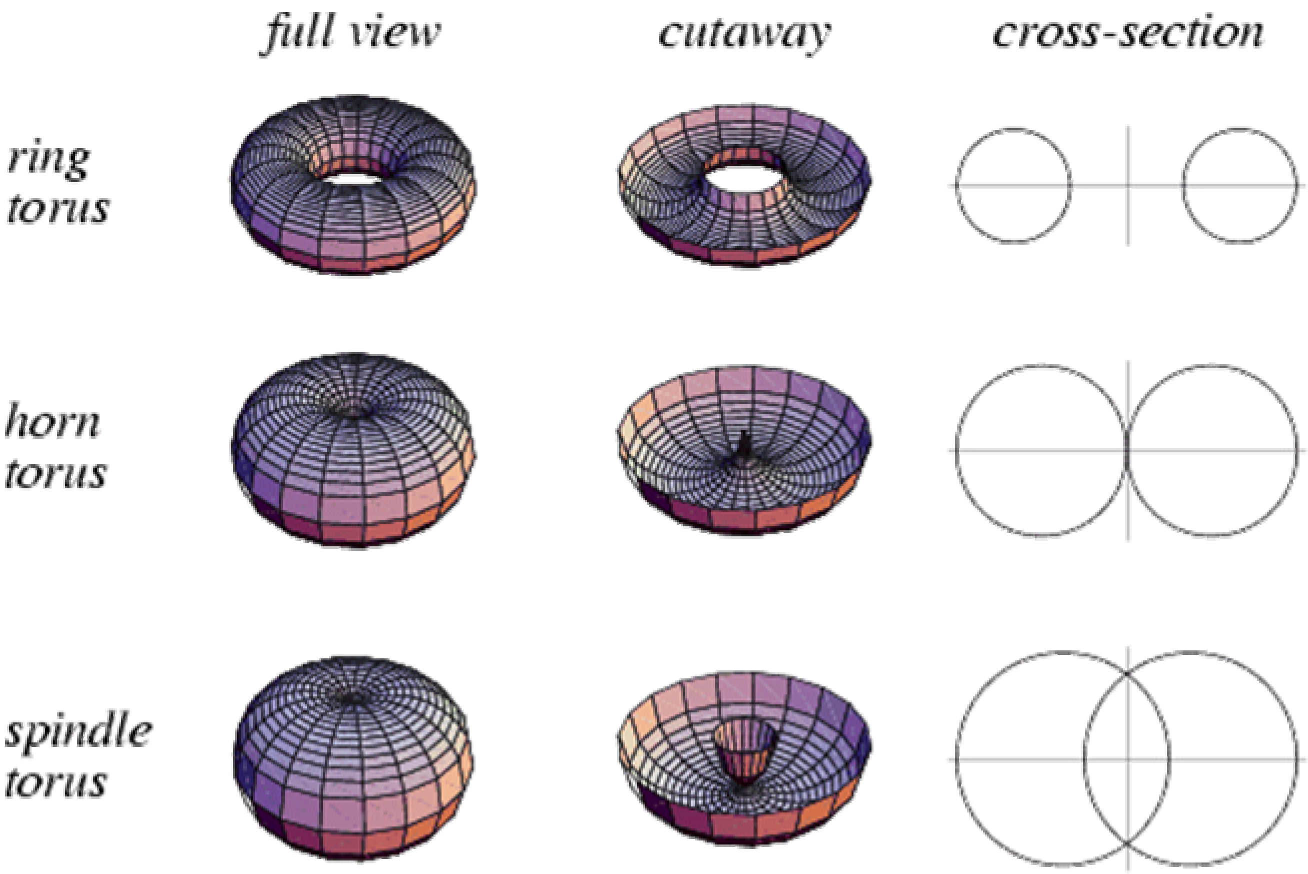


Torus



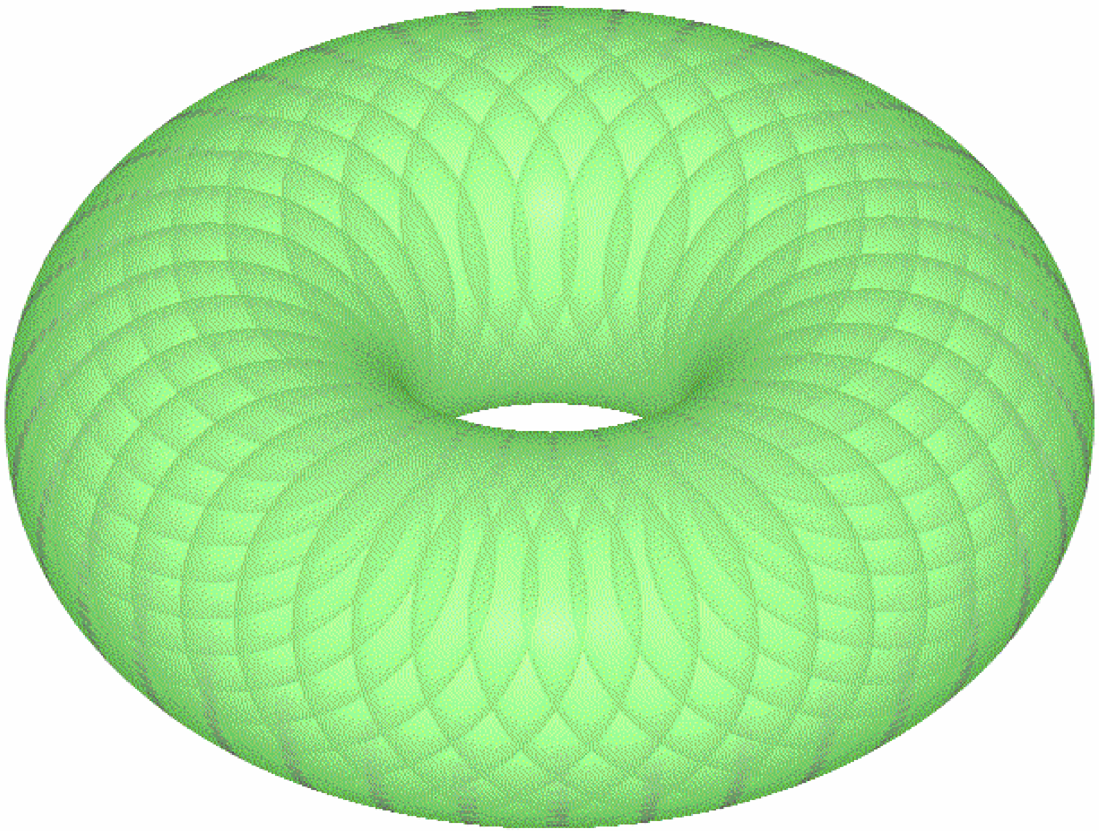
z (axis of rotation)
equatorial circle
meridian
circle
throat circle
y
x

Classification of Toruses

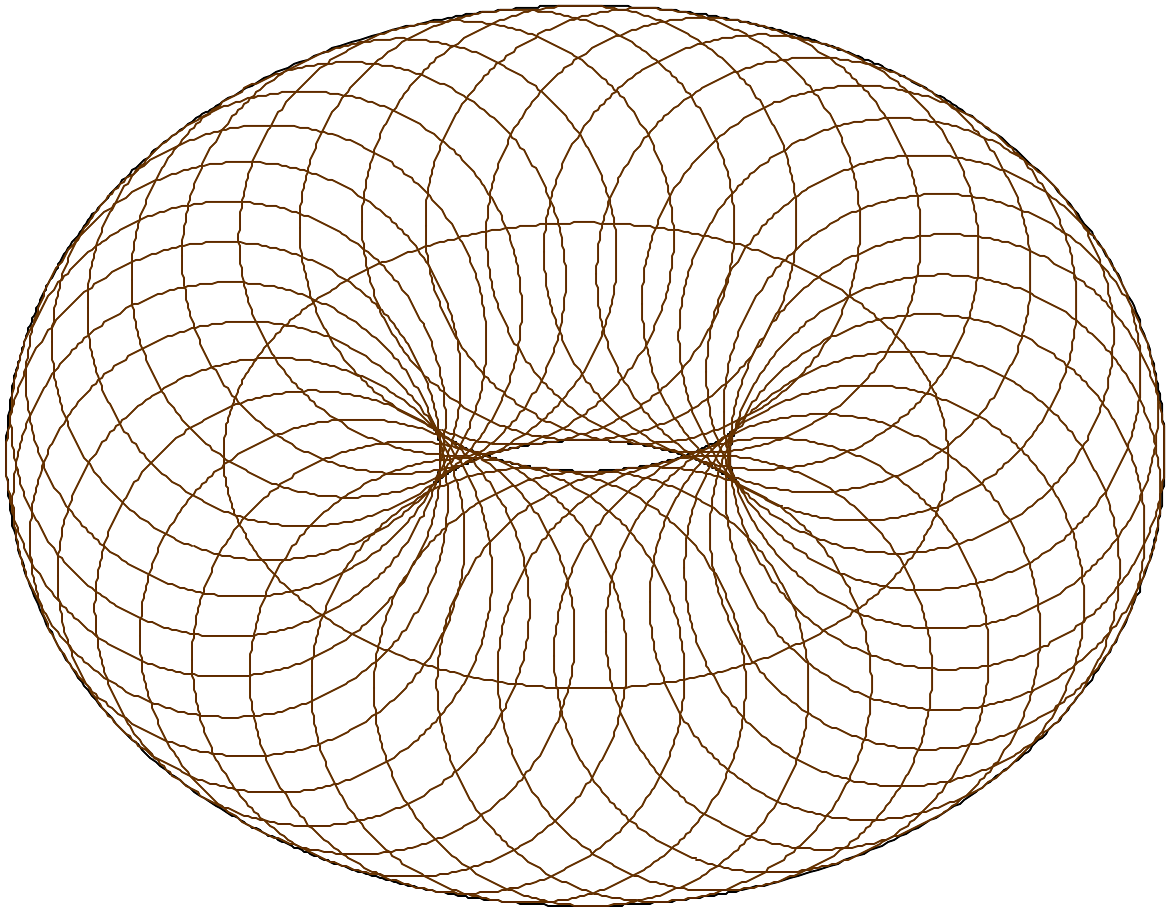


http://mathworld.wolfram.com/Torus.html

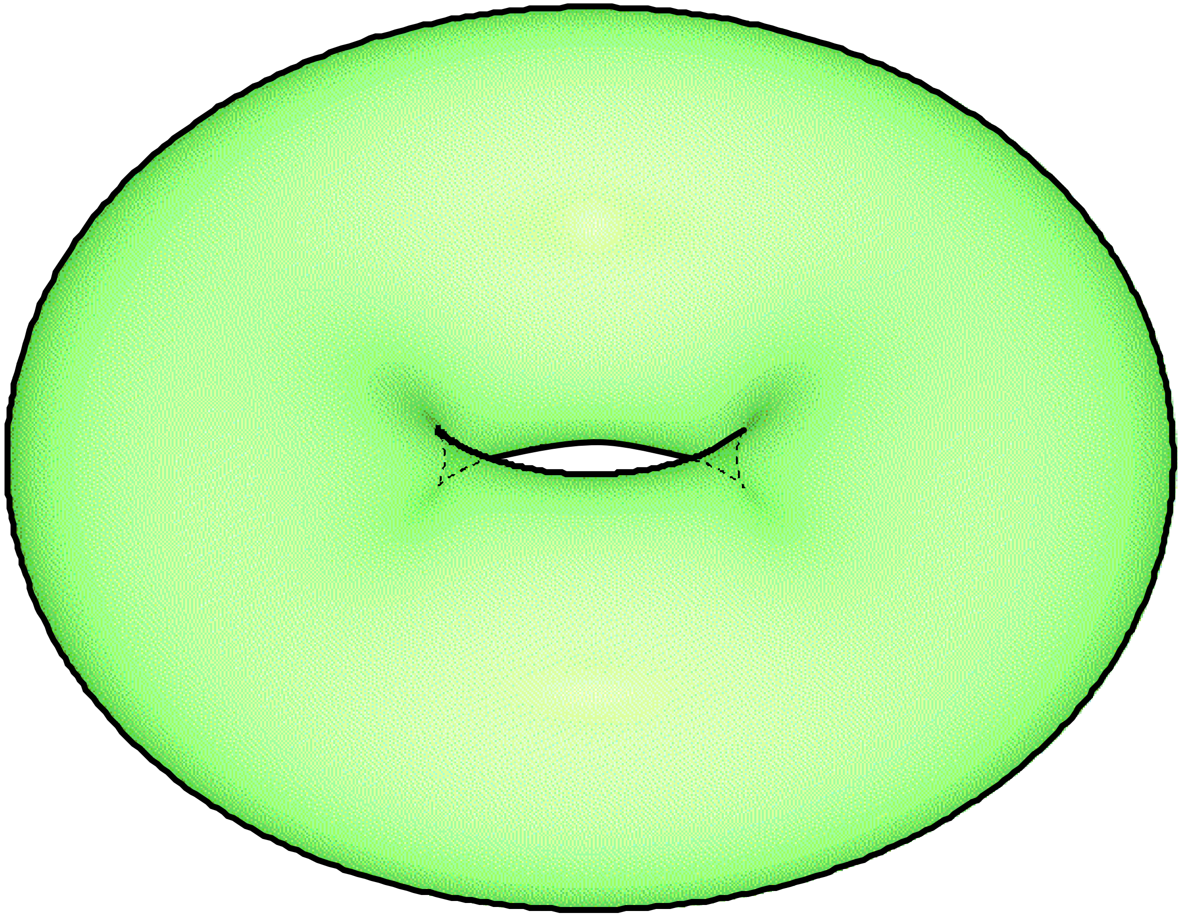
Torus as Envelope of Spheres



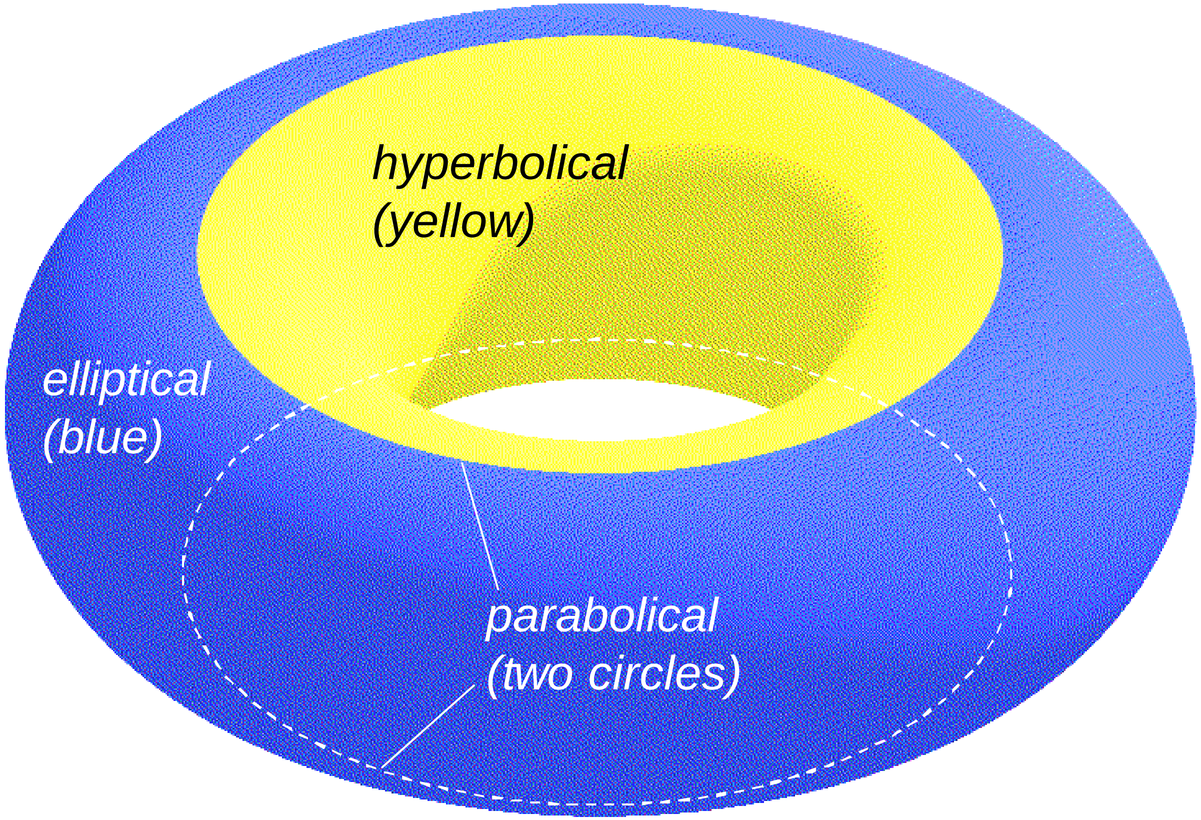
Outline of Torus as Envelope of Circles



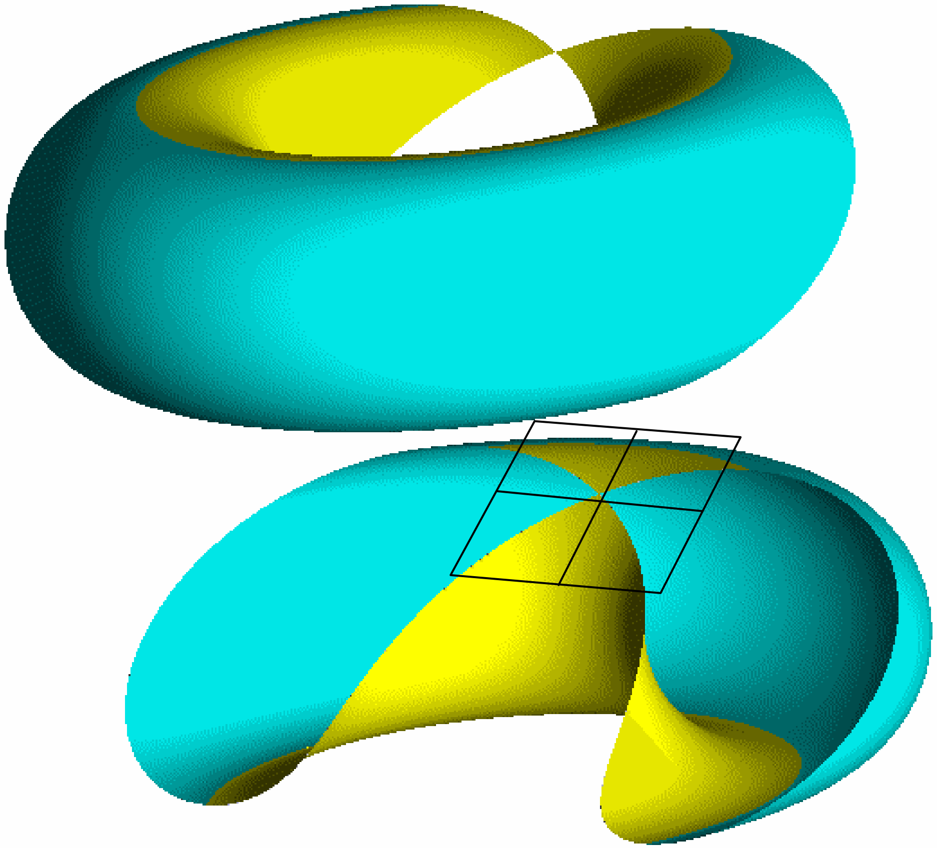
Outline of Torus



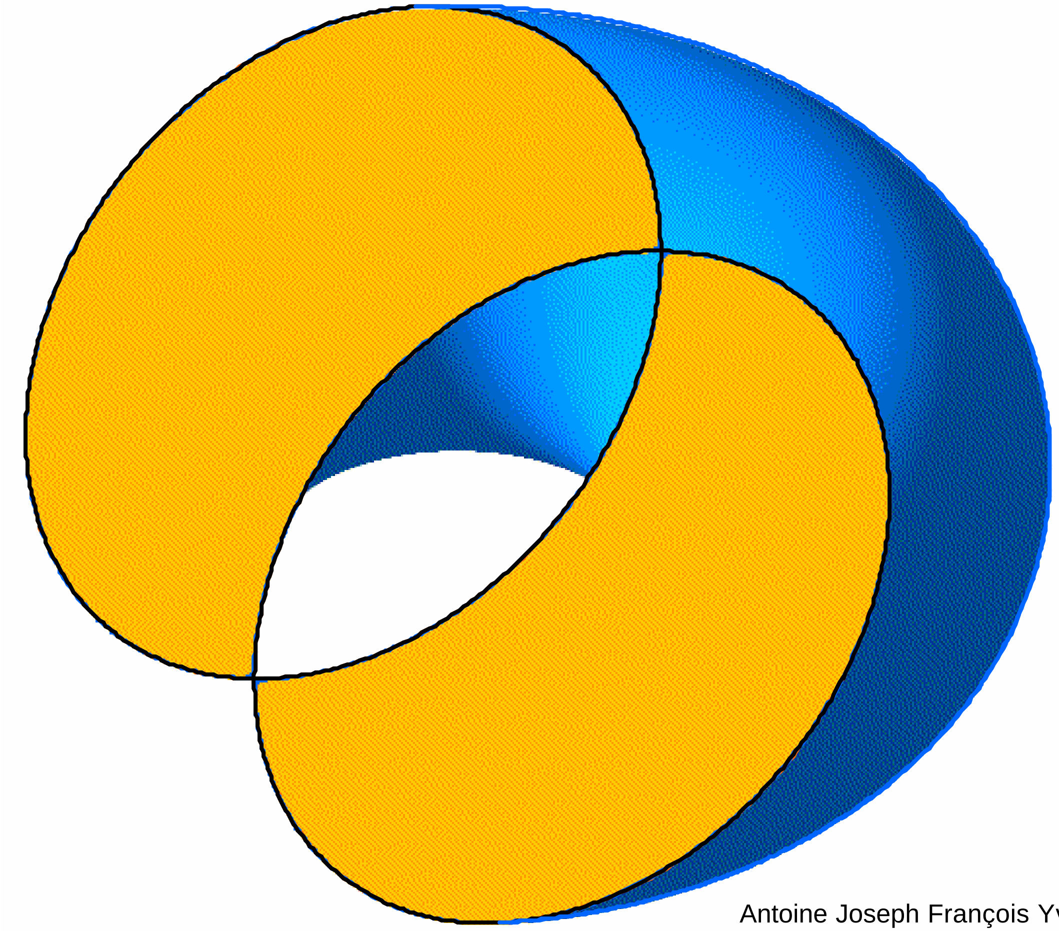
Classification of Points of Surface



Tangent plane at Hyperbolic Point

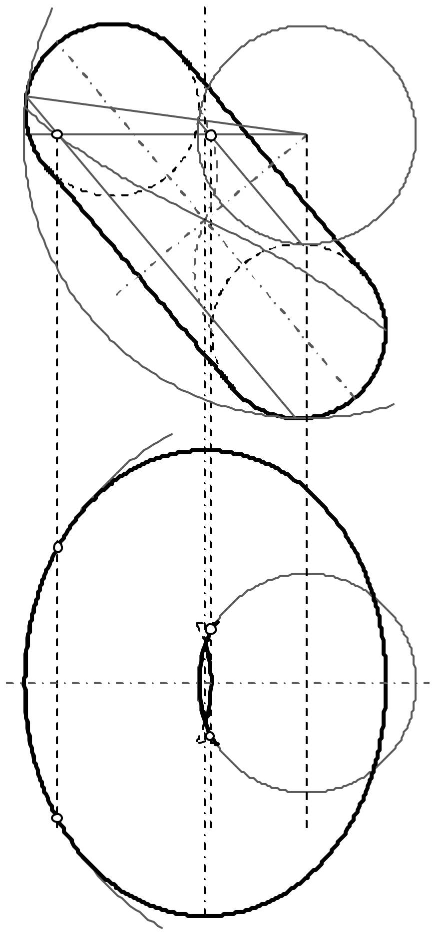


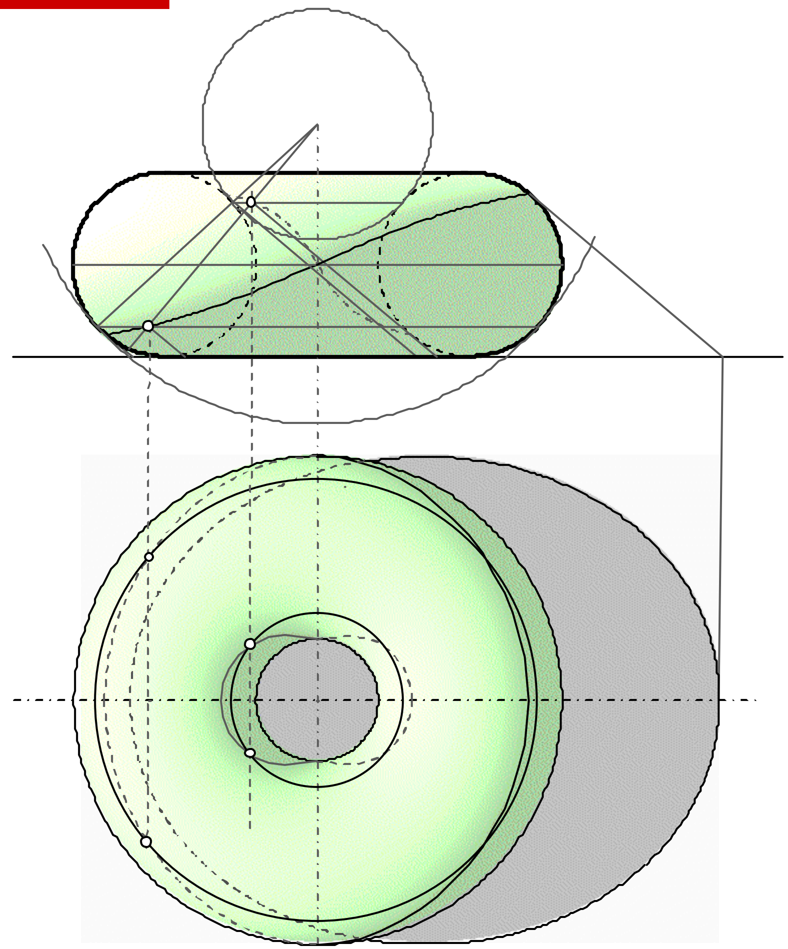
Villarceau Circles



Antoine Joseph François Yvon Villarceau (1813-1889)

Construction of Contour and Shadow





Ruled Surface

http://www.amsta.leeds.ac.uk/~khouston/ruled.htm

http://www.cs.mtu.edu/~shene/COURSES/cs3621/LAB/surface/ruled.html

http://mathworld.wolfram.com/RuledSurface.html

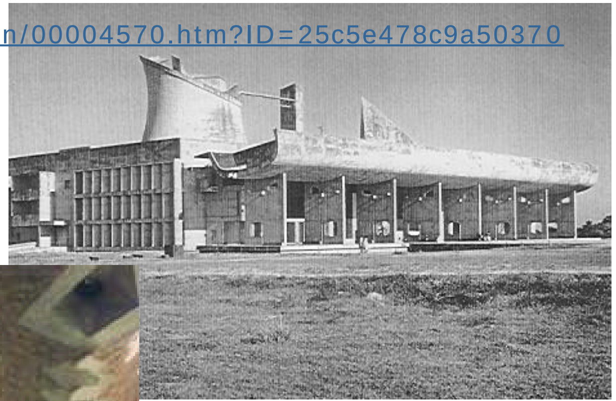
http://en.wikipedia.org/wiki/Ruled_surface

http://www.geometrie.tuwien.ac.at/havlicek/torse.html

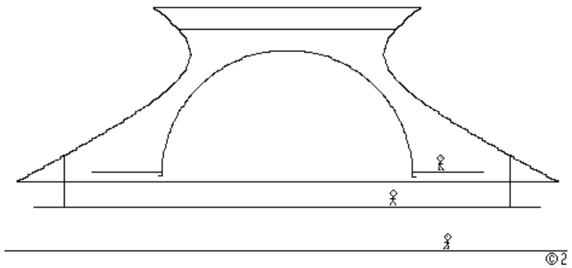
http://www.f.waseda.jp/takezawa/mathenglish/geometry/surface2.htm

Hyperboloid of One Sheet

http://www.archinform.net/medien/00004570.htm?ID=25c5e478c9a50370 03d984a4e4803ded



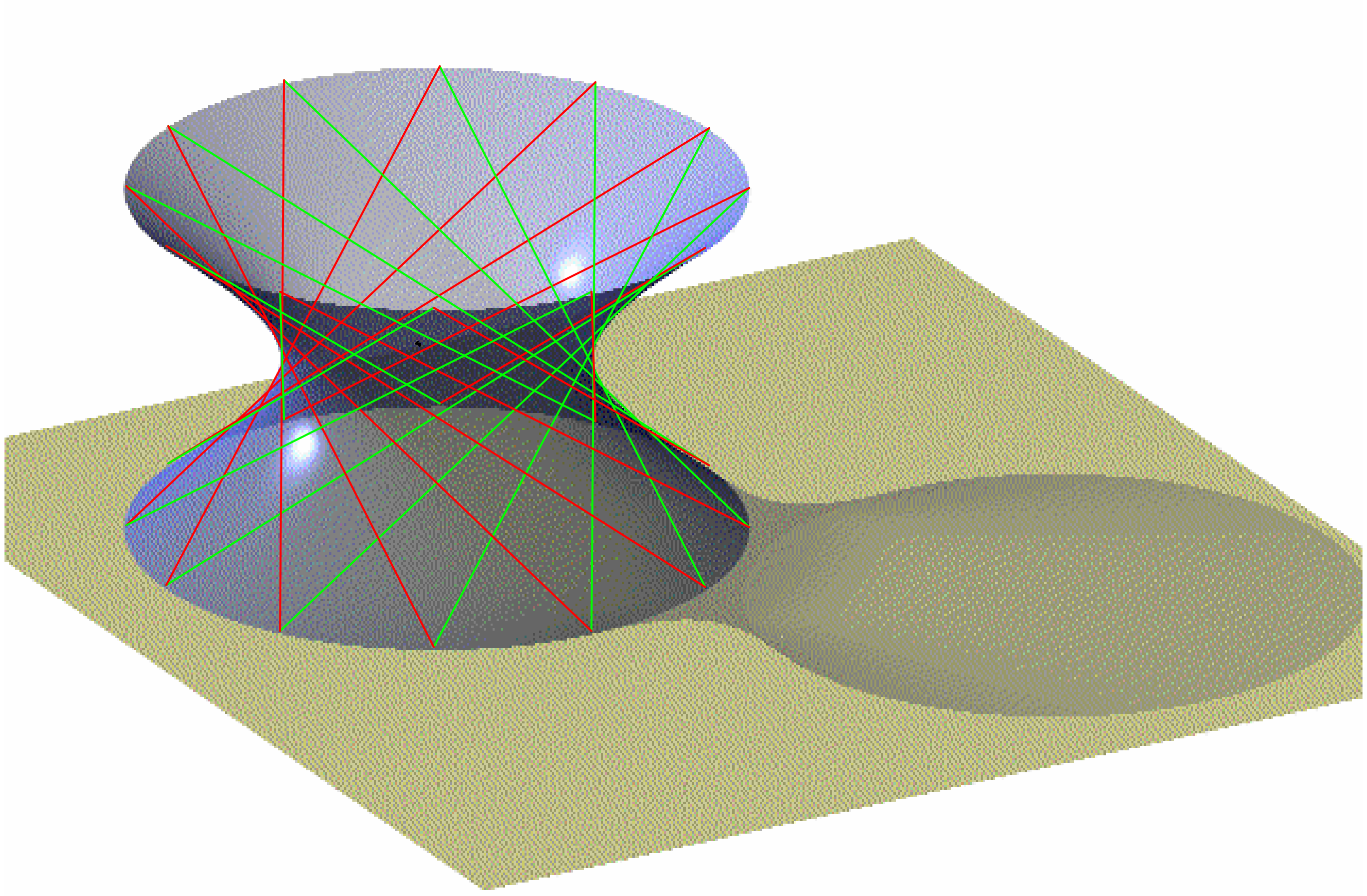
St. Louis Science Center Planetarium



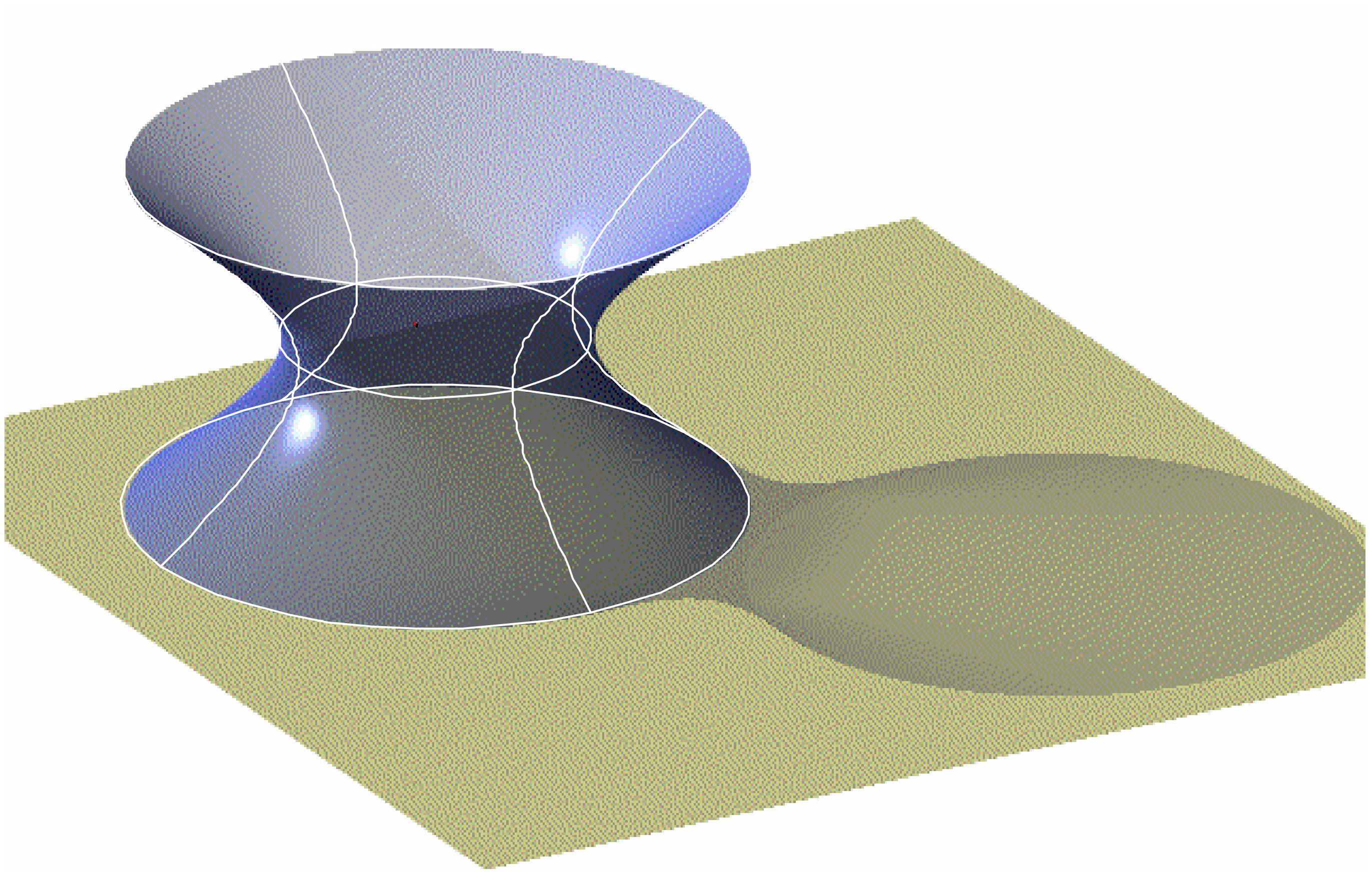
http://www.jug.net/wt/slscp/slscpa.htm

http://www.earth-auroville.com/index.php?nav=menu&pg=vault&id1=18

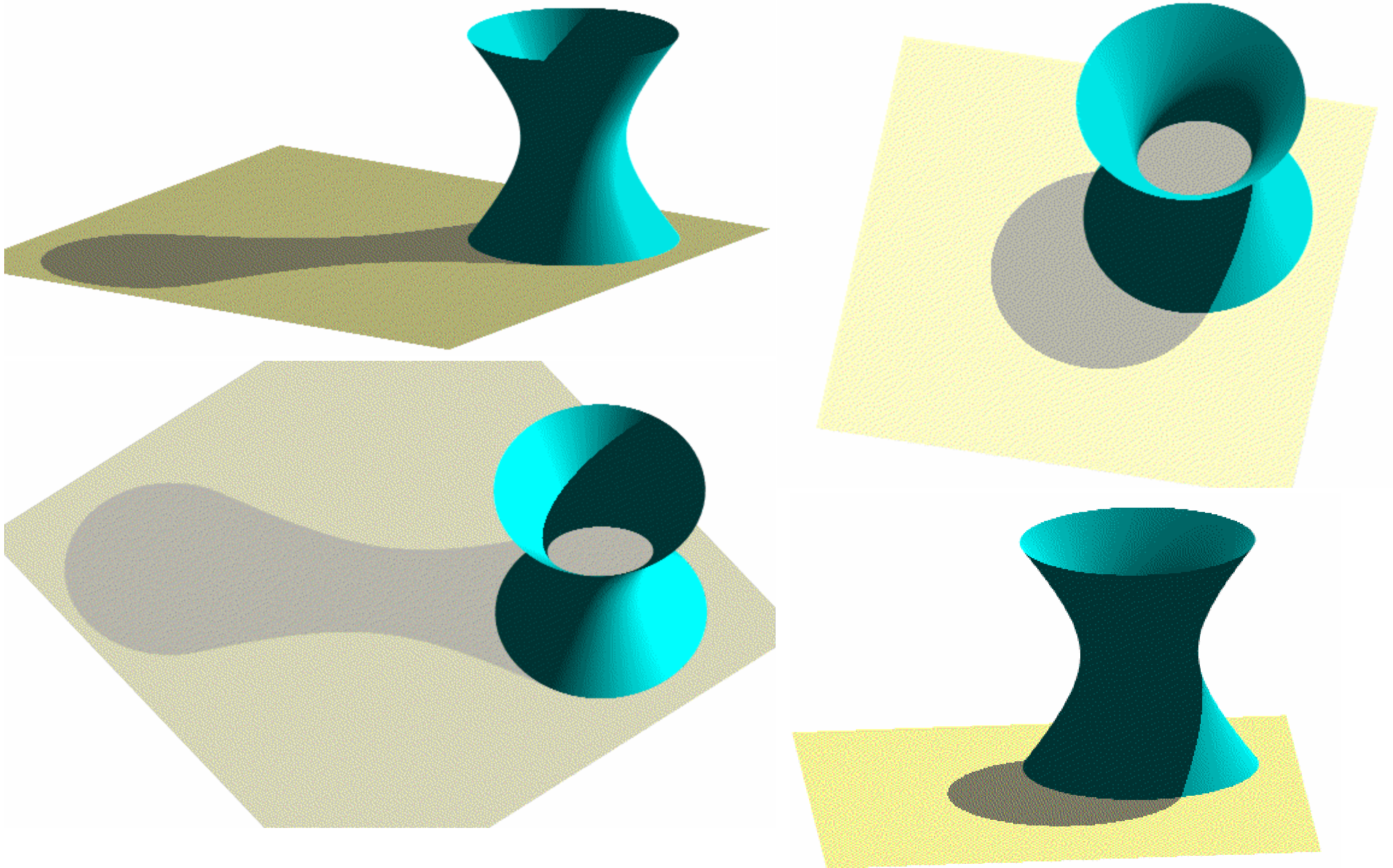
Hyperboloid of One Sheet



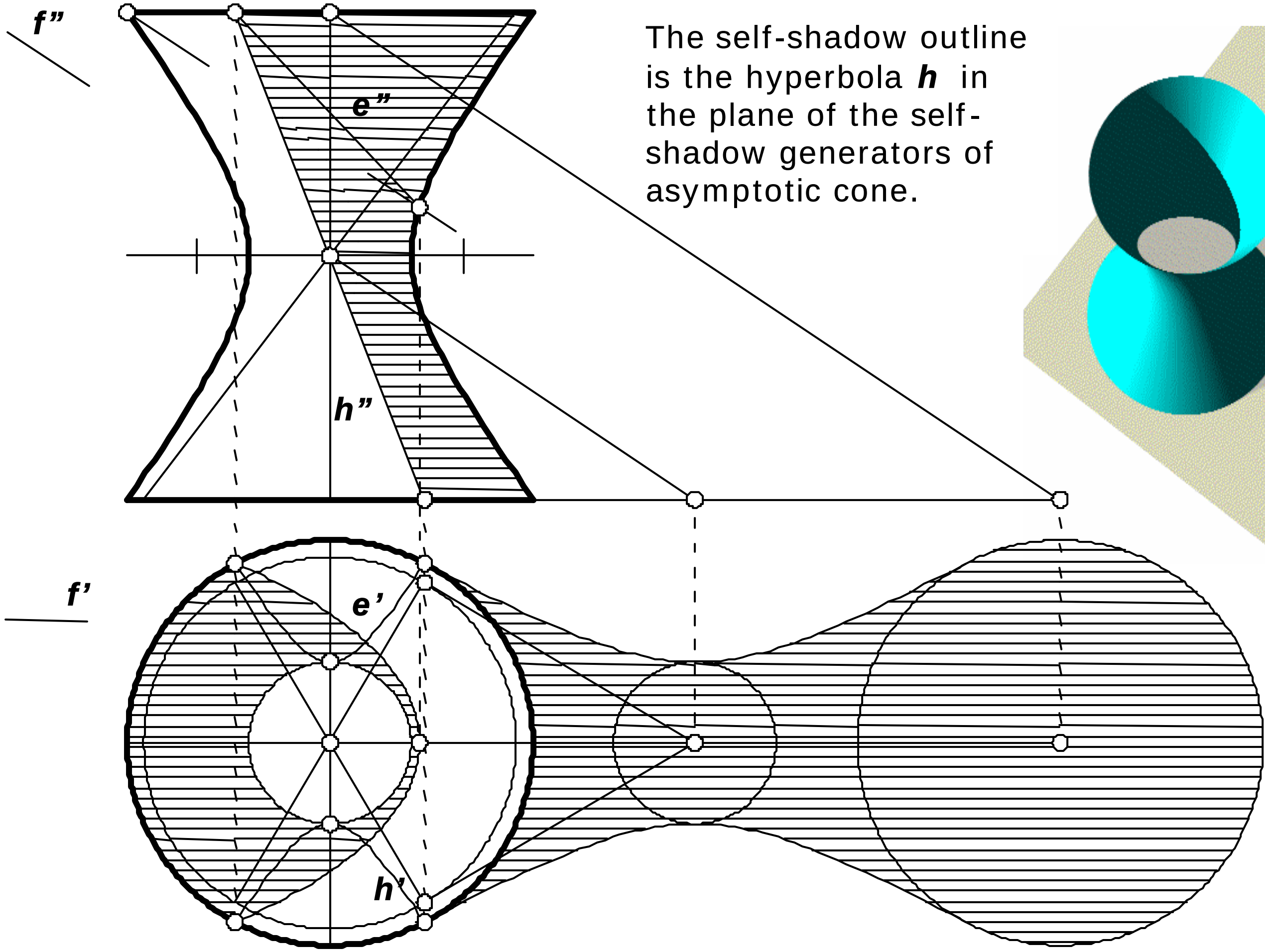
Hyperboloid of One Sheet, Surface of Revolution



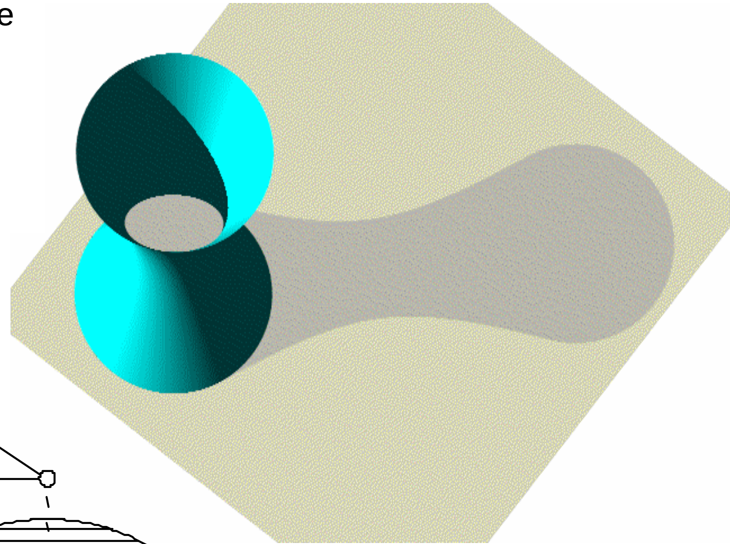
Shadows on Hyperboloid of one Sheet



Hyperboloid of One Sheet, Shadow 1

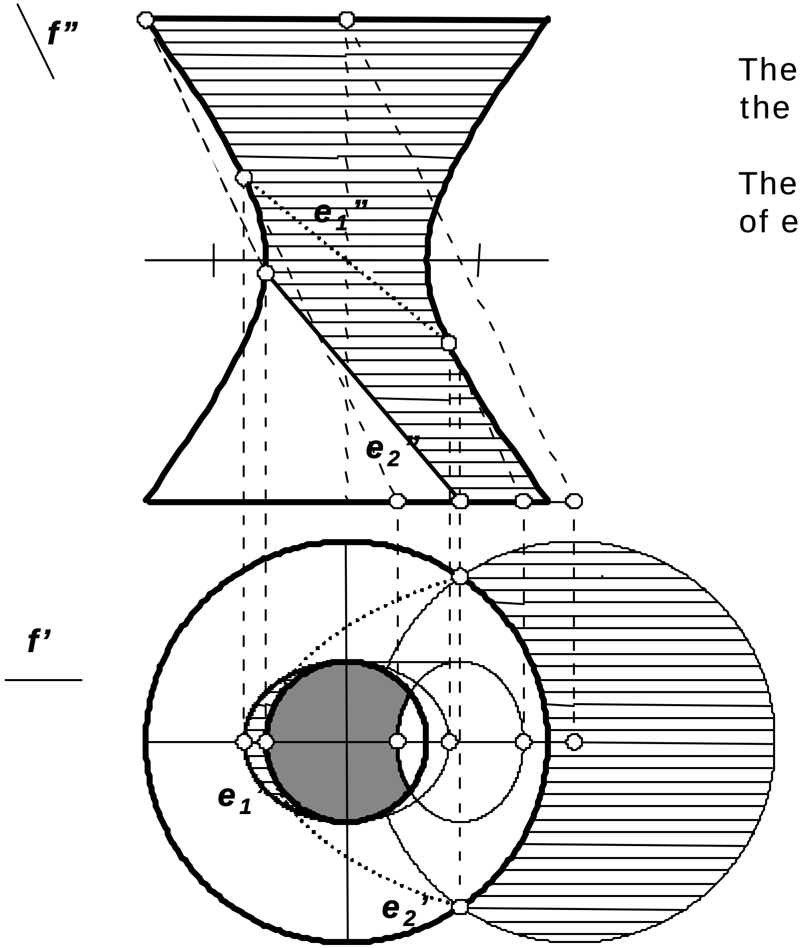


The self-shadow outline is the hyperbola ***h*** in the plane of the self-shadow generators of asymptotic cone.



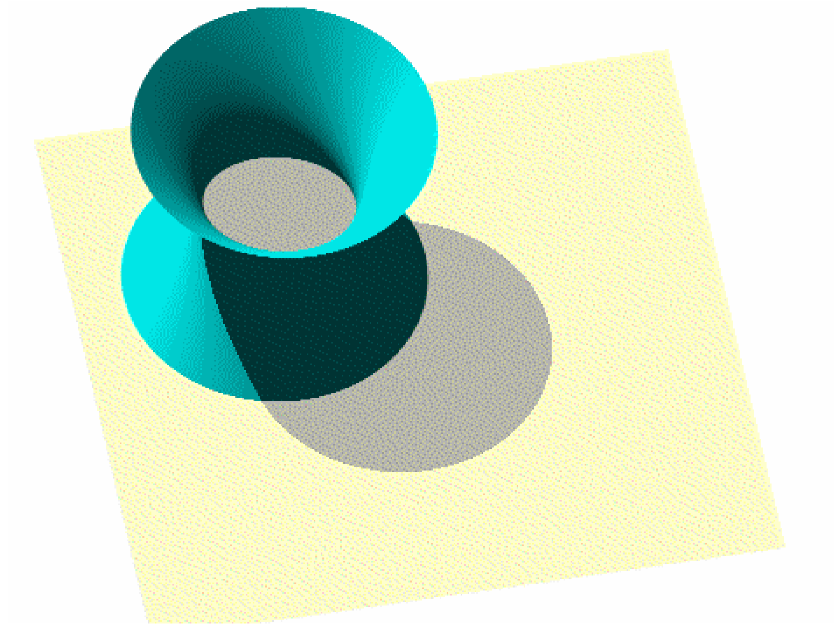
The cast-shadow on the hyperboloid itself is an arc of ellipse ***e*** inside of the surface.

Hyperboloid of One Sheet, Shadow 2

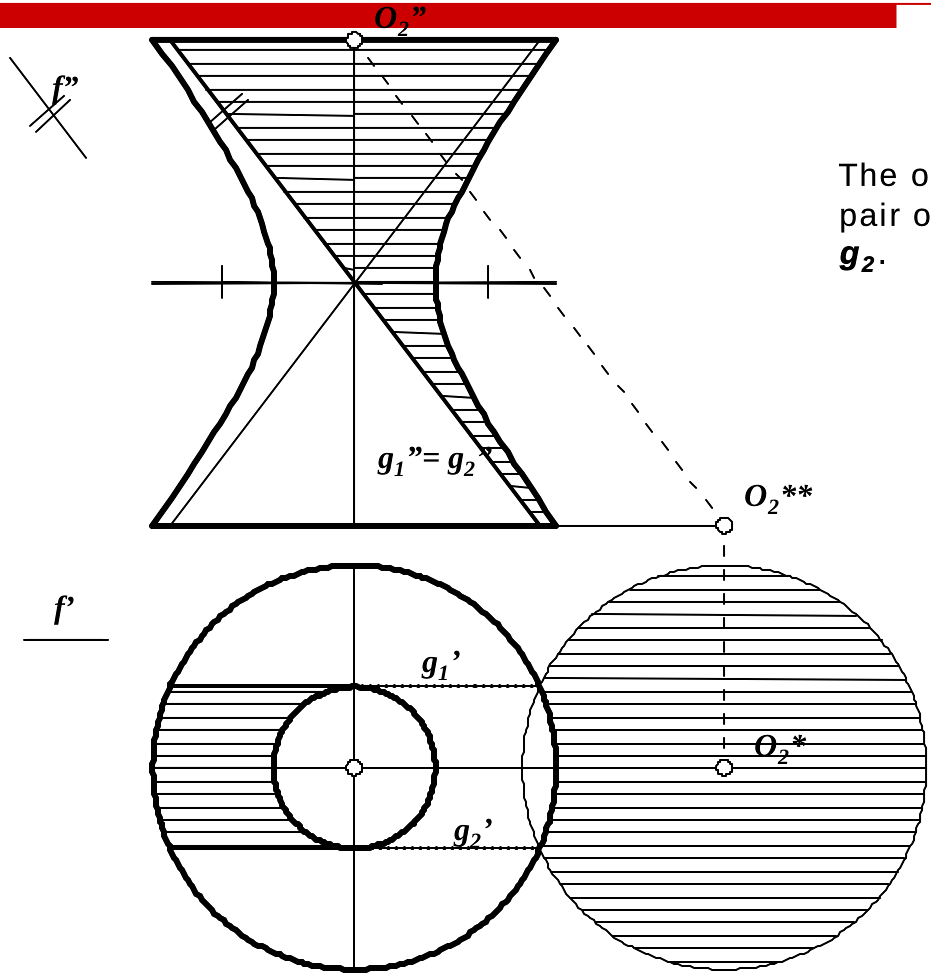


The self-shadow outline is the ellipse e_1 inside of the surface.

The cast-shadow on the hyperboloid itself is an arc of ellipse e_2 on the outher side of the surface.

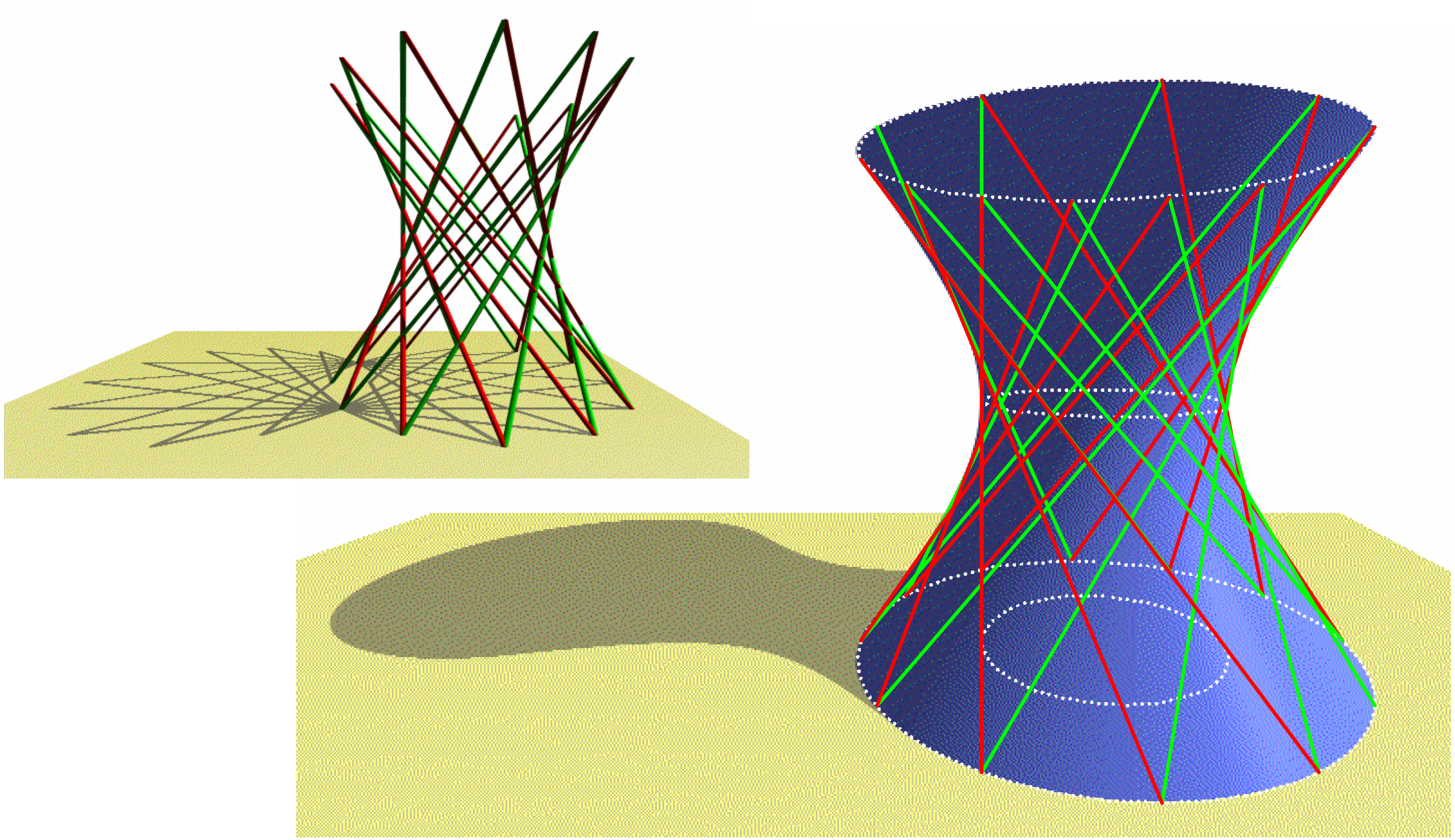


Hyperboloid of One Sheet, Shadow 3

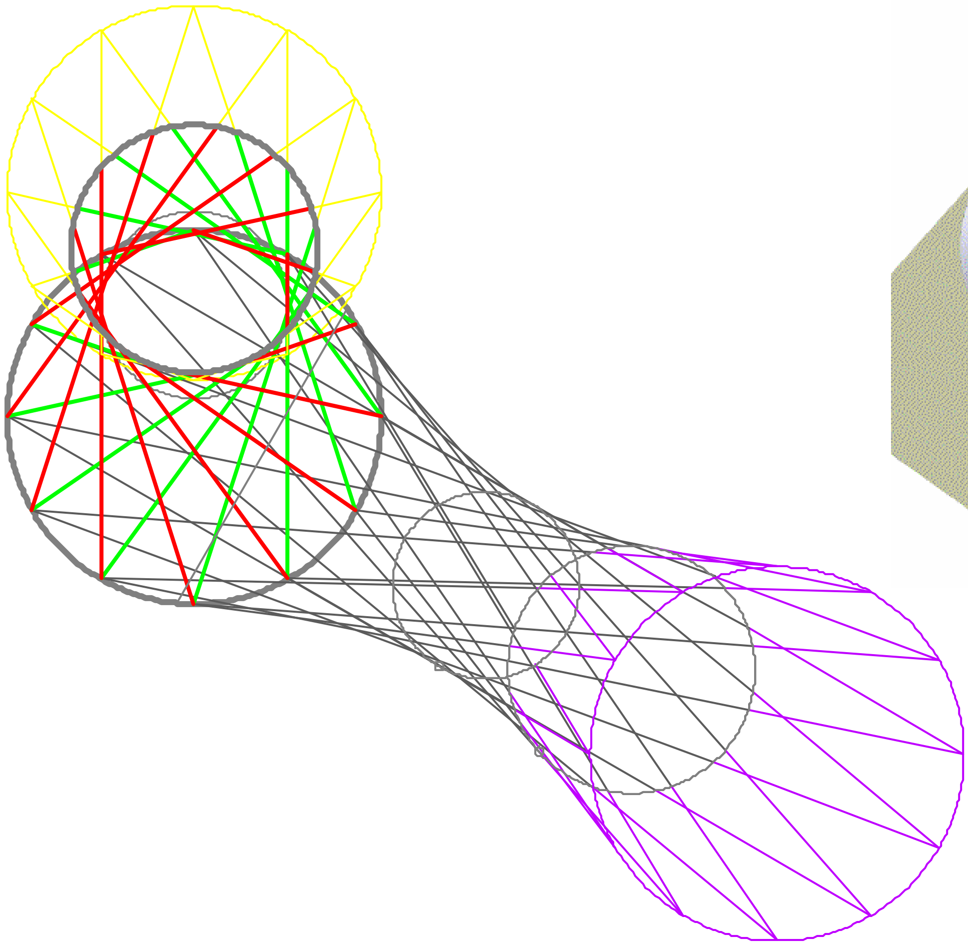


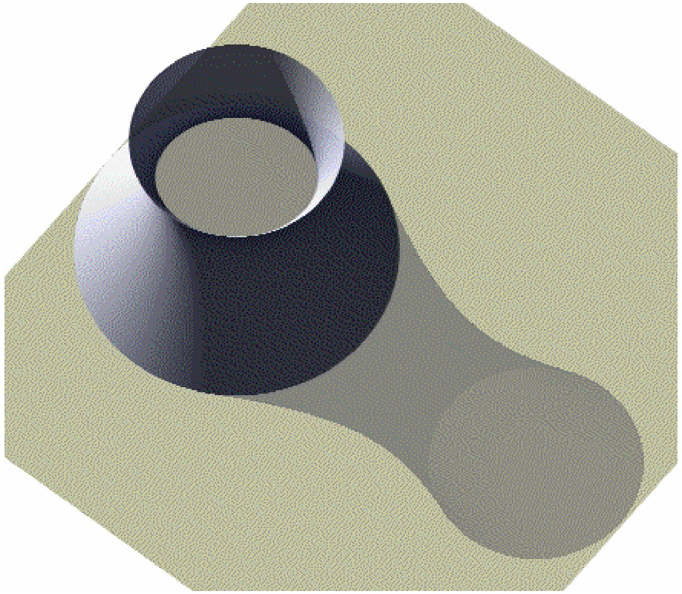
The outline of the self_shadow is a pair of parallel generators $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$.

Hyperboloid of One Sheet in Perspective

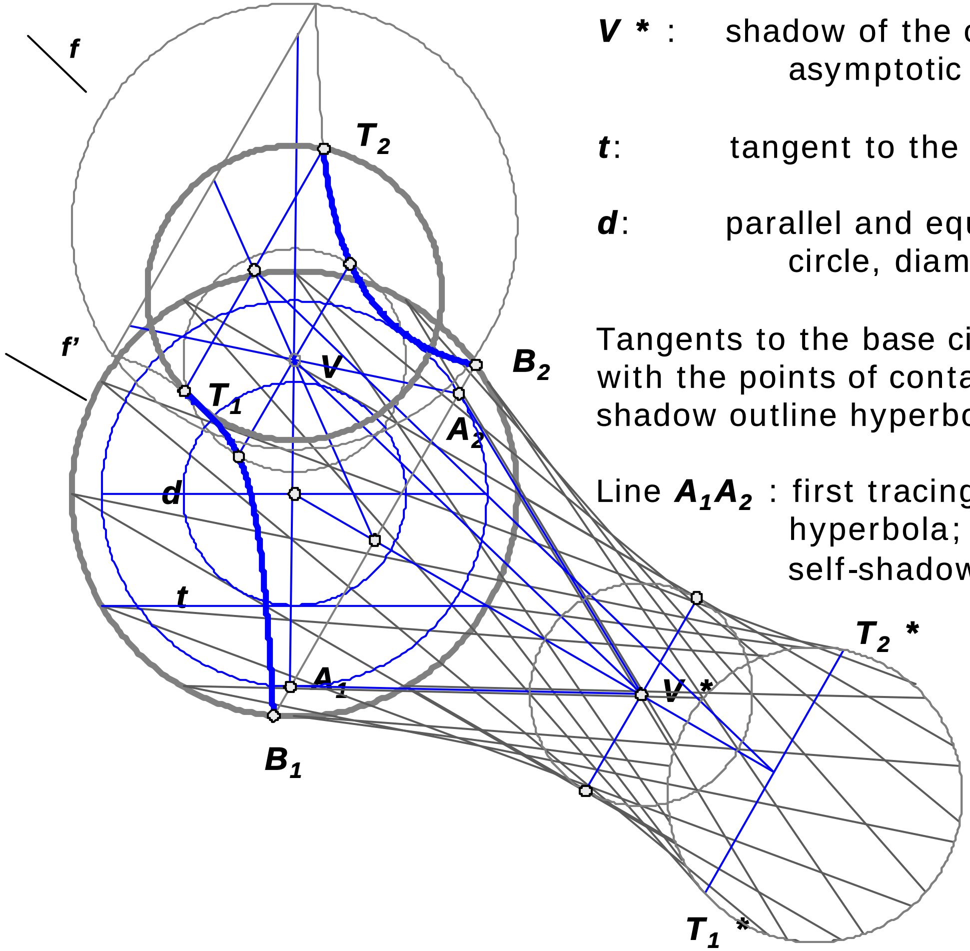


Hyperboloid in Military Axonometry





Construction of Self-shadow an Cast Shadow



V * : shadow of the center ***V*** that is the vetrex of the asymptotic cone

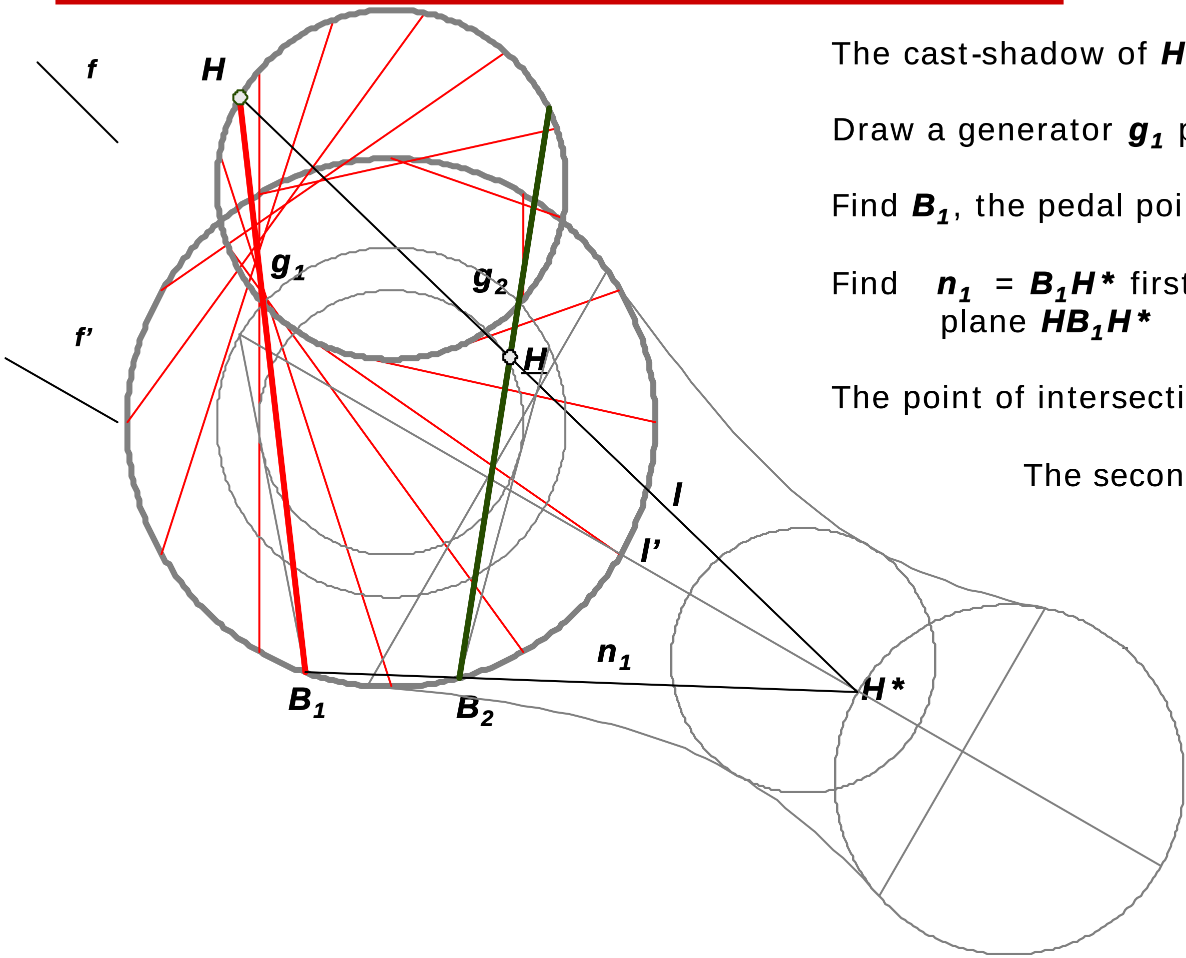
t: tangent to the throat circle, chord of the base circle

d: parallel and equal to ***t*** through the center of the base circle, diameter of the asymptotic cone

Tangents to the base circle of the asymptotic cone from ***V**** with the points of contact A_1 and A_2 : asymptotes of the cast-shadow outline hyperbola

Line A_1A_2 : first tracing line of the plane of self-shadow hyperbola; $V A_1$ and VA_2 : asymptotes of the self-shadow outline hyperbola

Construction of Projected Shadow



The cast-shadow of ***H*** on the ground plane: ***H****

Draw a generator g_1 passing through ***H***

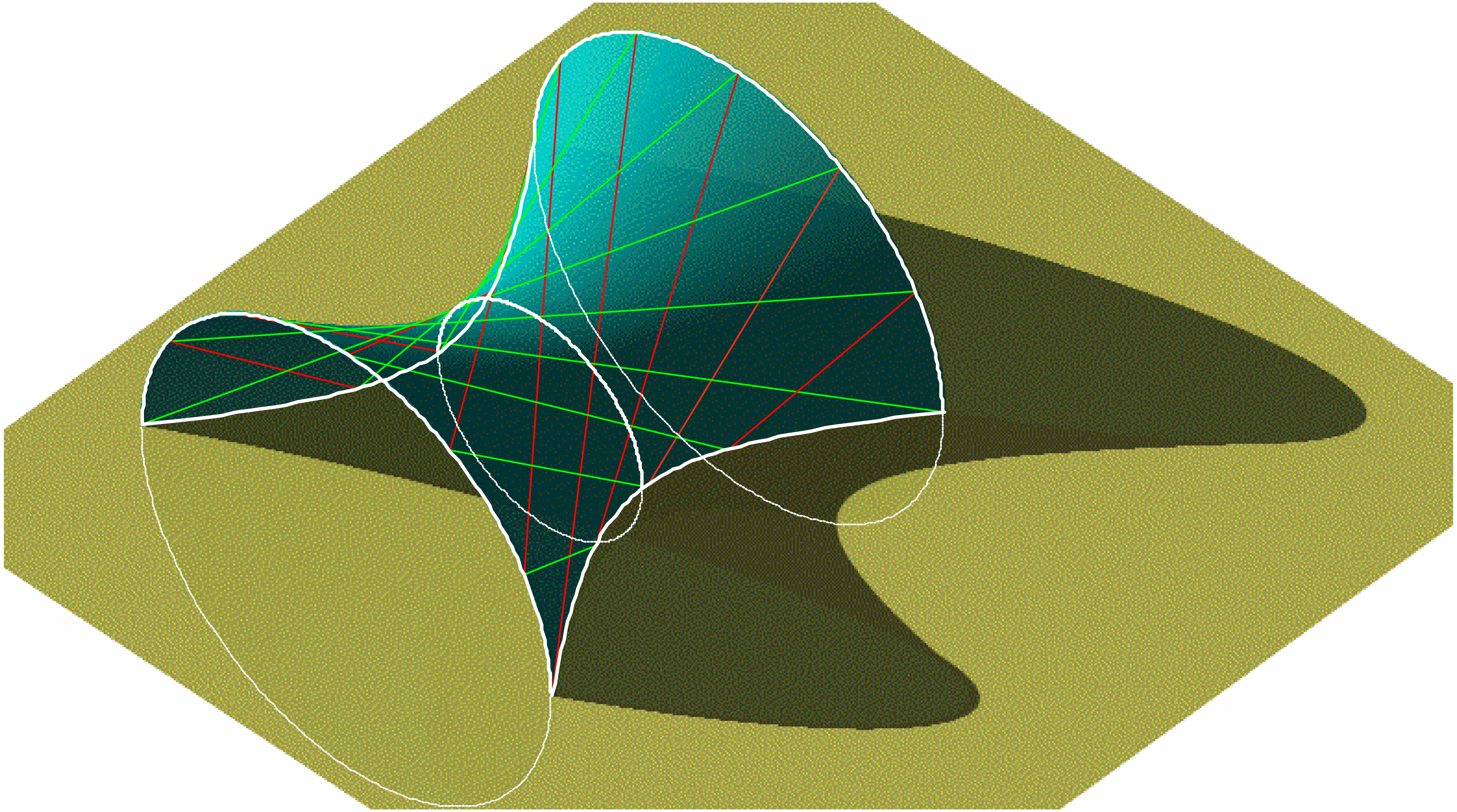
Find B_1, the pedal point of the generator g_1

Find $n_1 = B_1H^*$ first tracing line of the auxiliary plane HB_1H^*

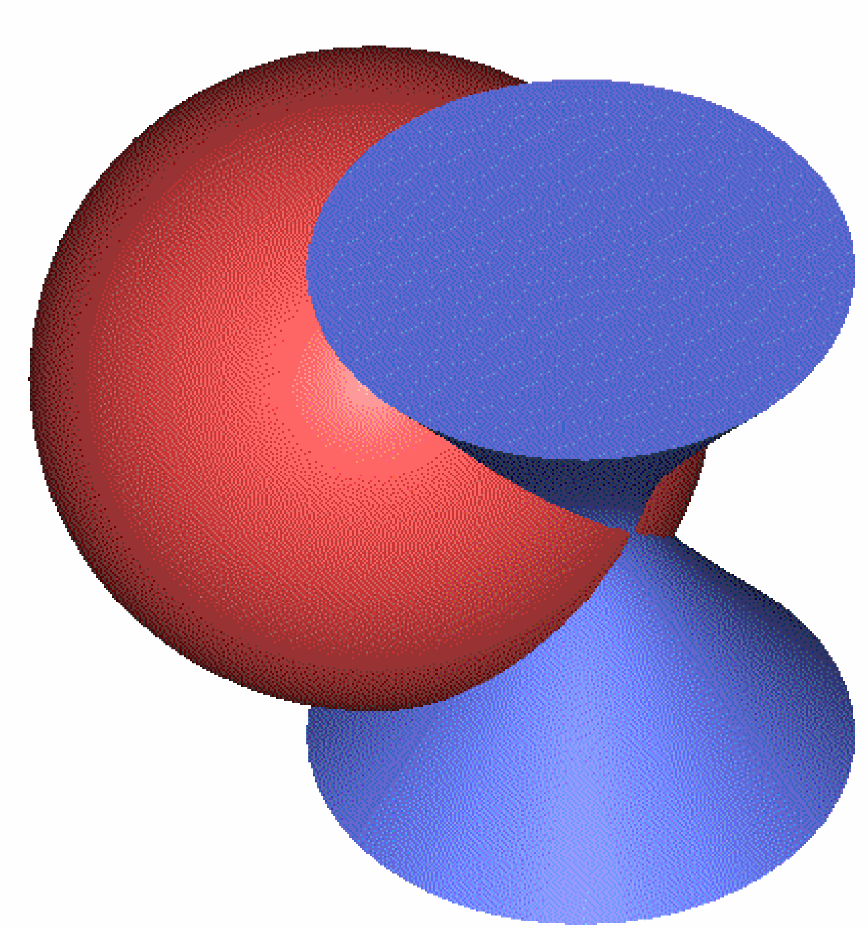
The point of intersection of the base circle and n_1 is B_2

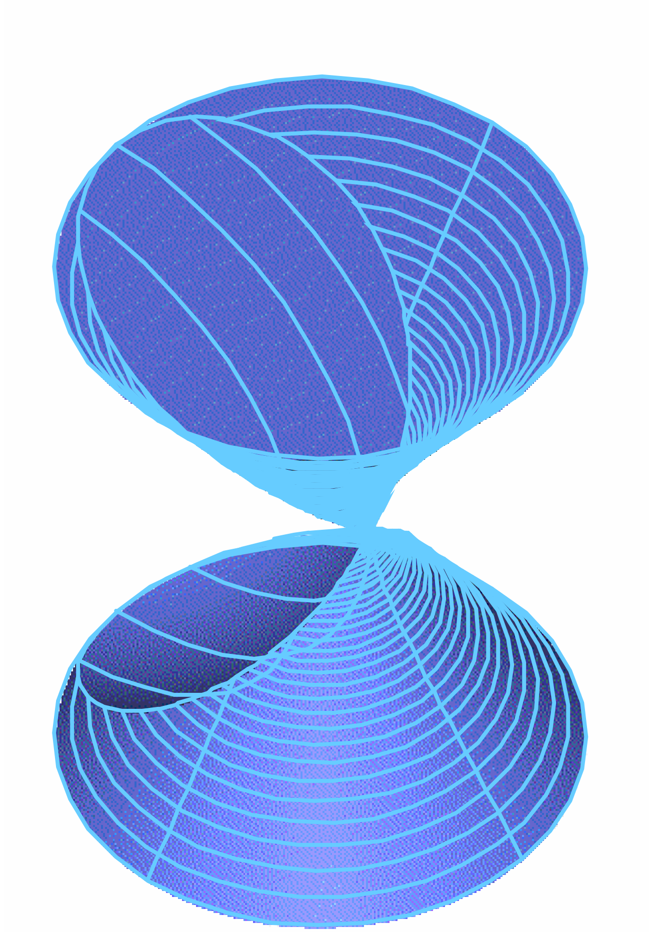
The second generator g_2 lying in the plane HB_1H^* intersects the ray of light ***l*** at $\underline{H}$, the lowest point of the ellipse, i.e. the outline of the projected shadow inside.

Hyperboloid of One Sheet with Horizontal Axis



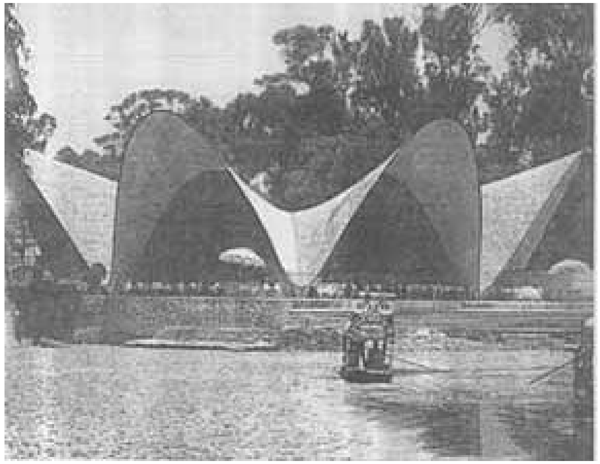
Hyperboloid of One Sheet, Intersection with Sphere



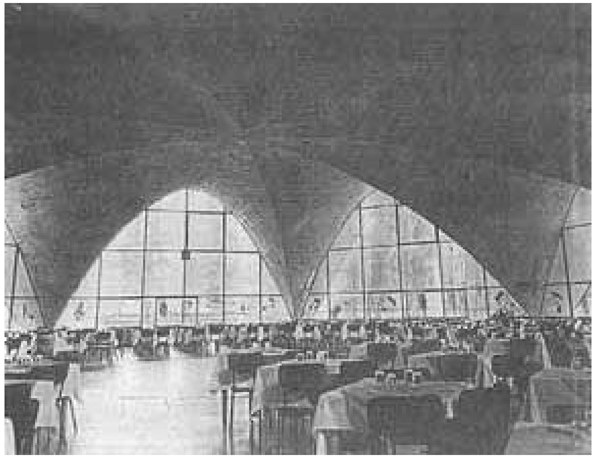


Ruled Surface: Hyperbolic Paraboloid

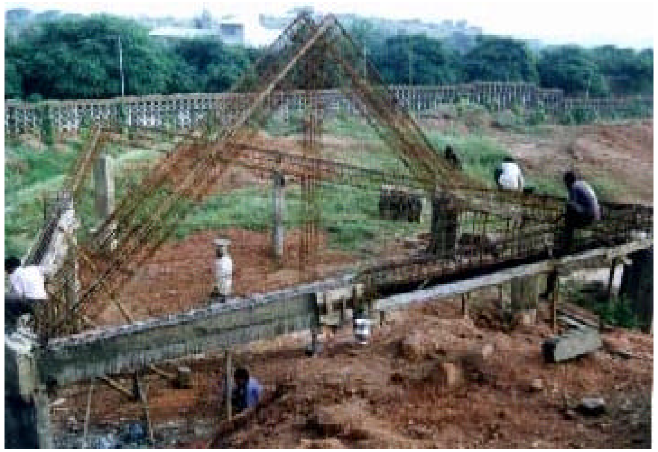
http://www.recentpast.org/types/hyperpara/index.html



http://www.ketchum.org/shellpix.html#airform

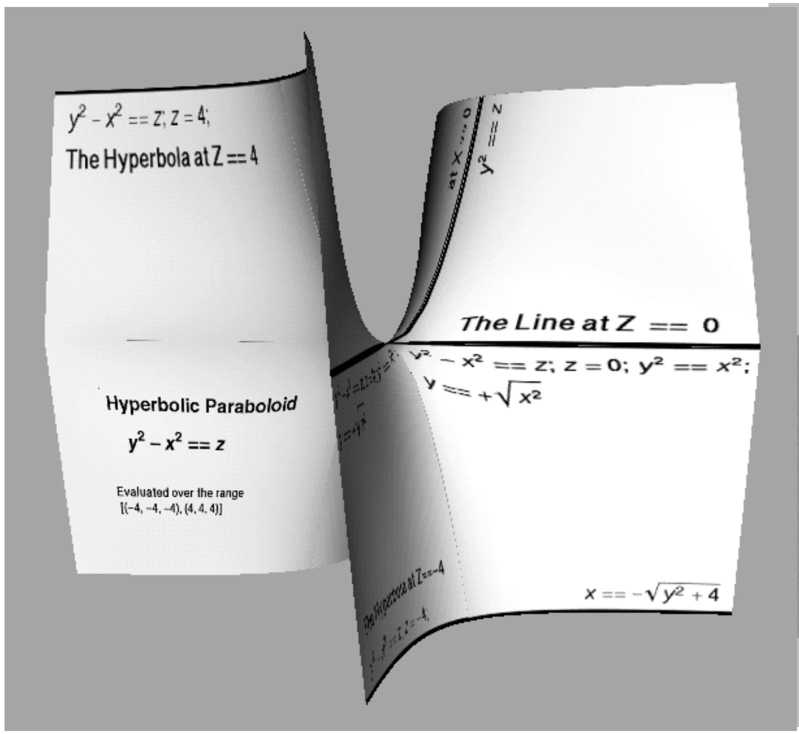


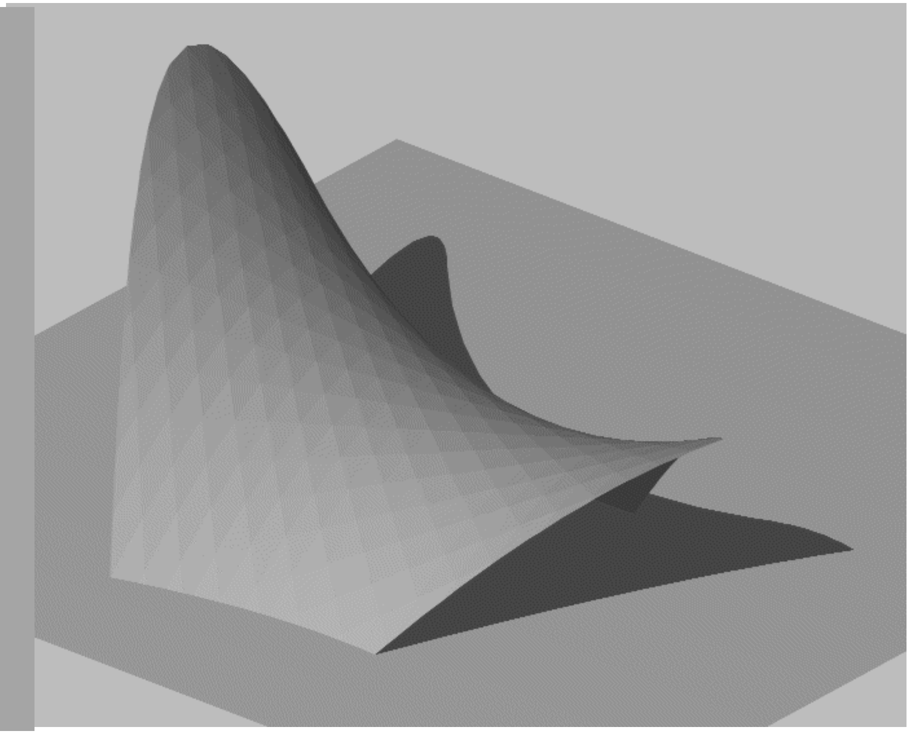
Hyperbolic Paraboloid: Construction



http://www.anangpur.com/struc7.html

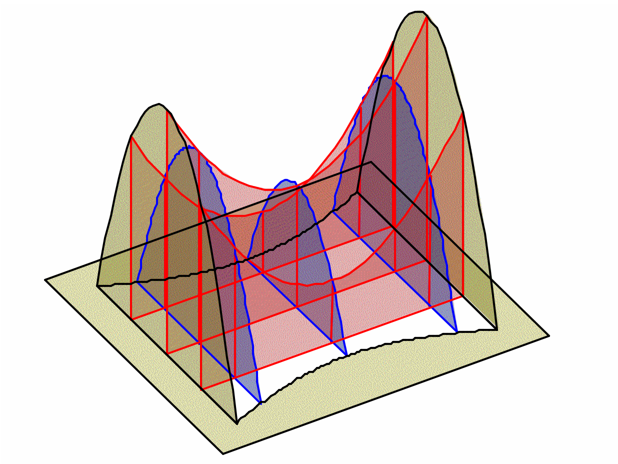
Saddle Surface

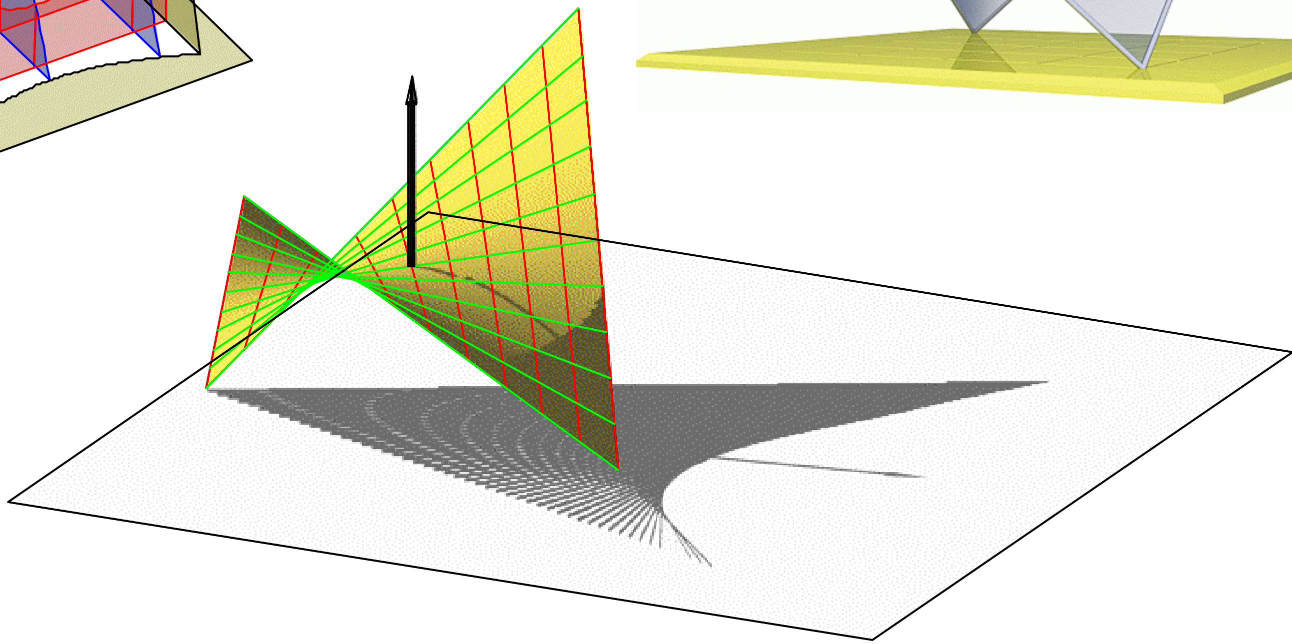


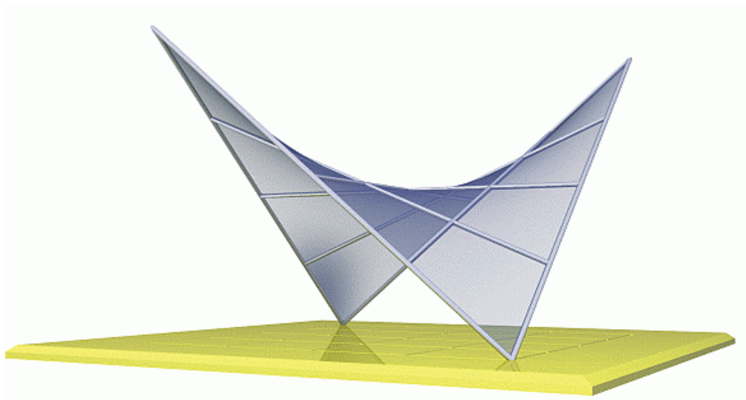


http://emsh.calarts.edu/~mathart/Annotated_HyperPara.html

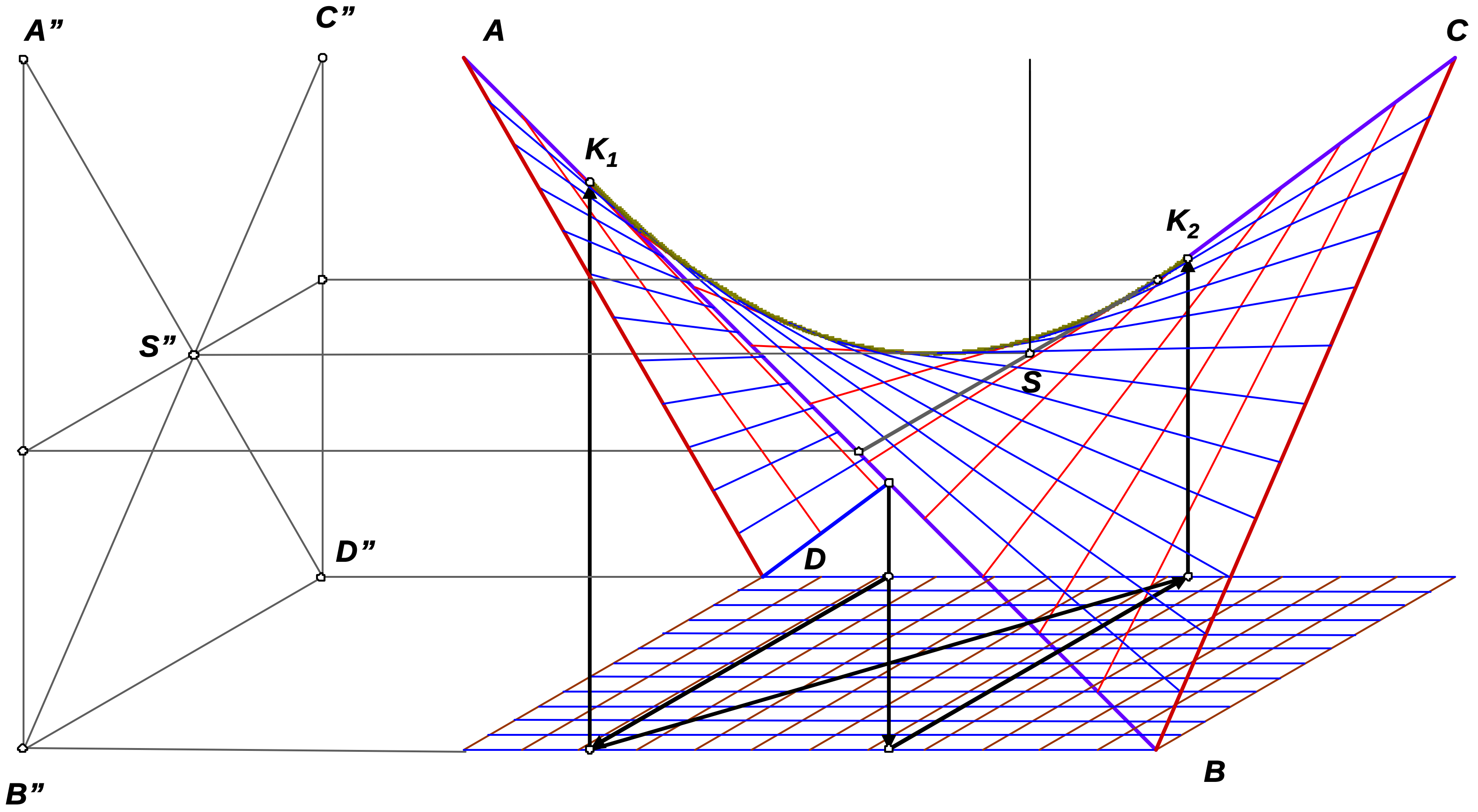
Axonometry and Perspective



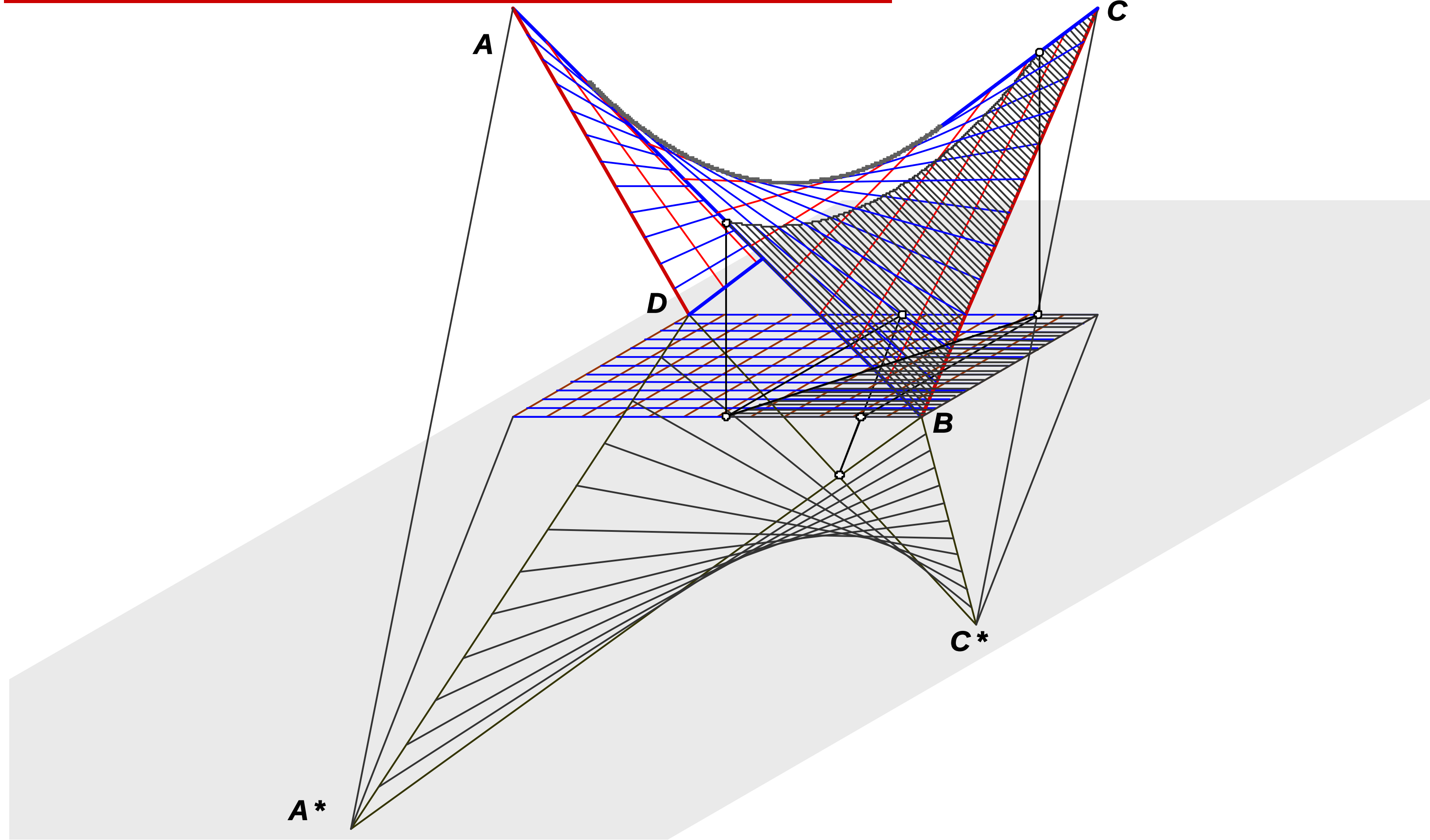




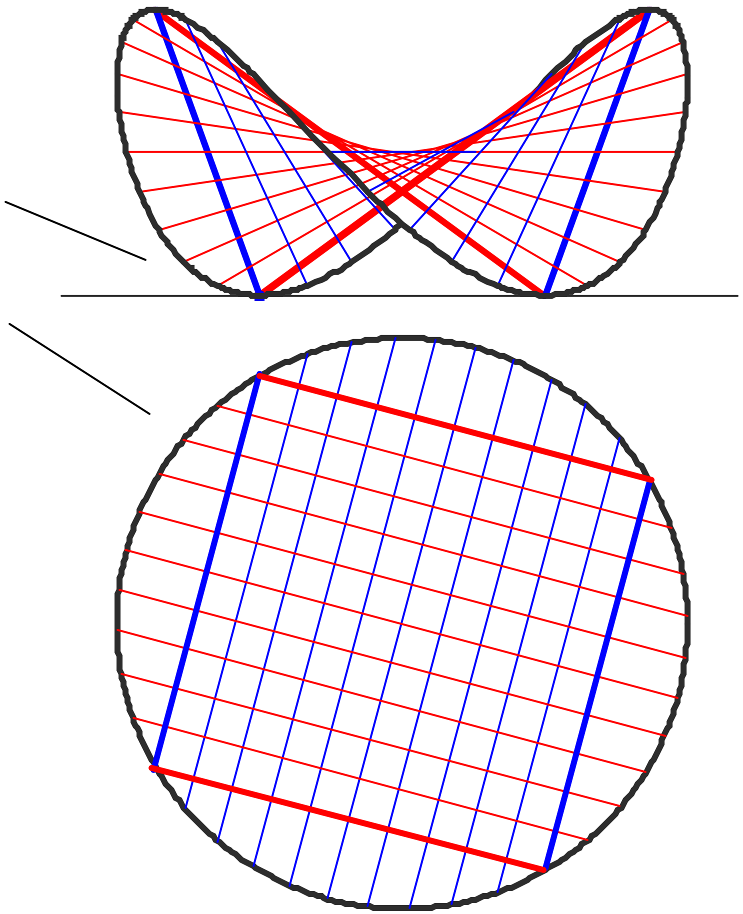
Saddle Point an Contour

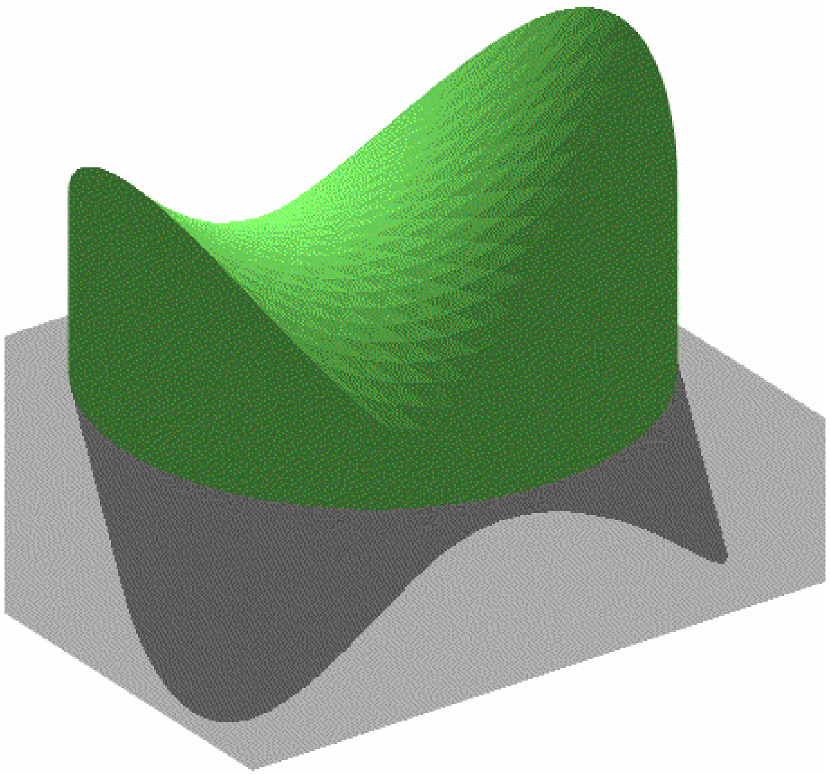


Shadow at Parallel Lighting

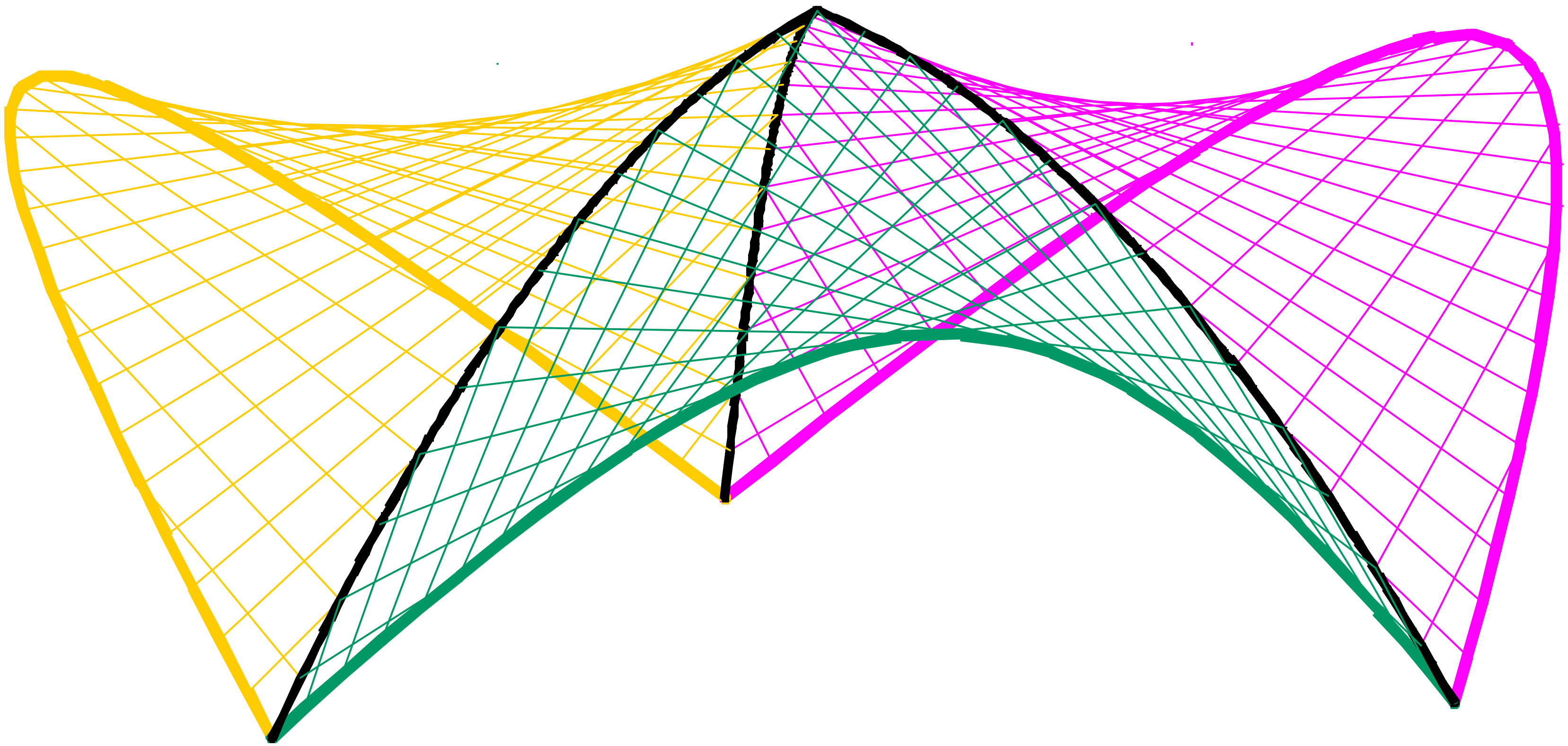


Intersection with Cylinder

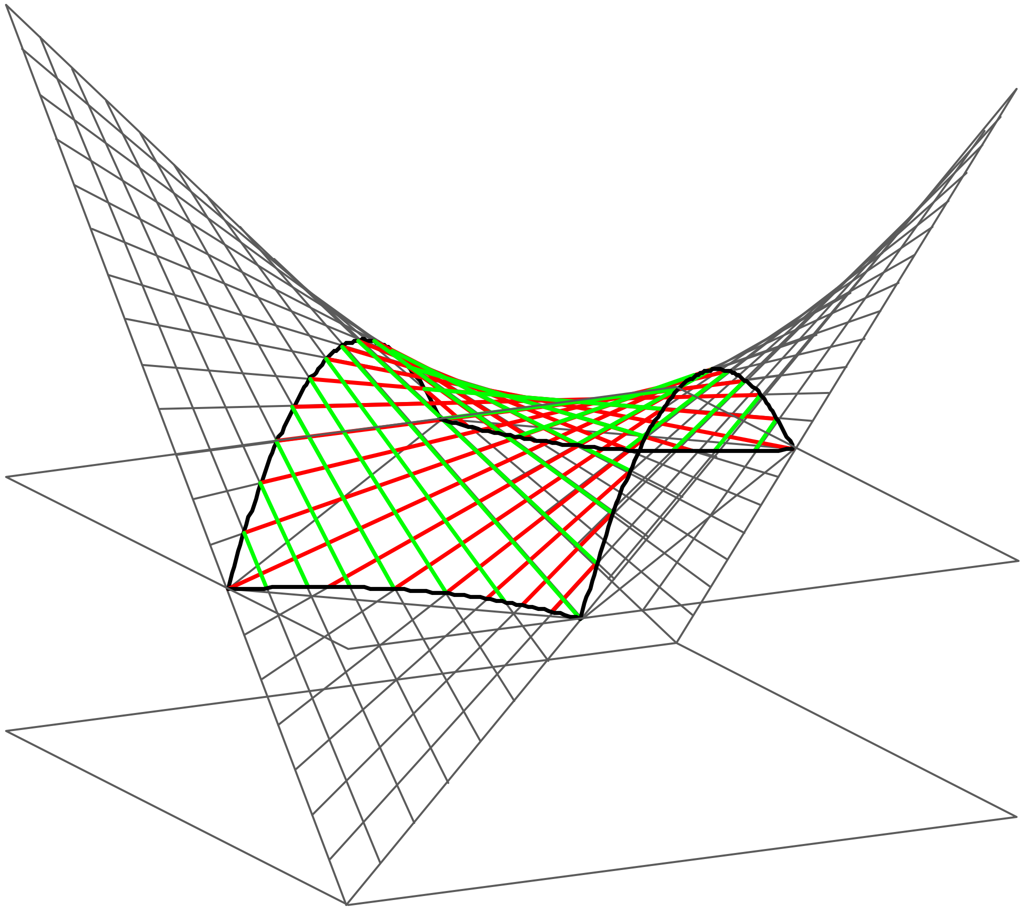




Composite Surface



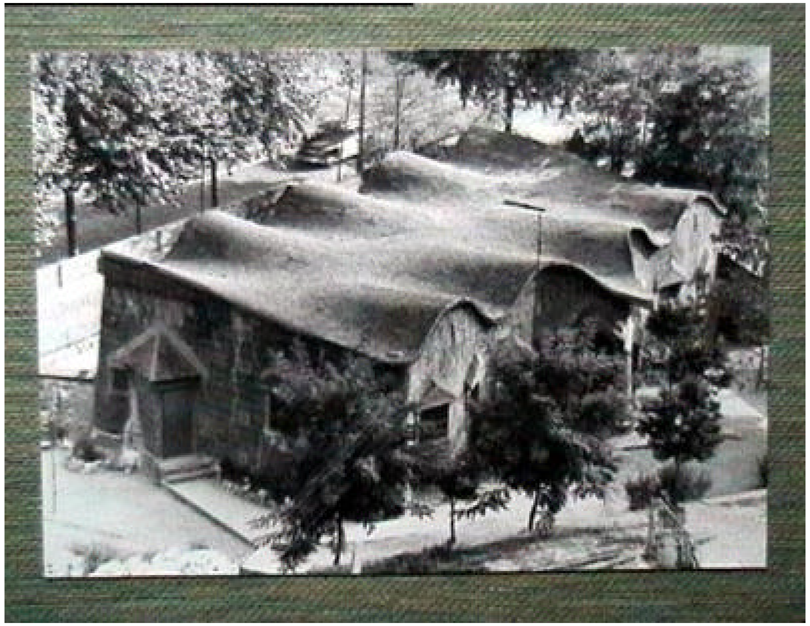
Intersection with Plane



Conoid

Conoid Studio, Interior.
Photo by Ezra Stoller (c)ESTO
Courtesy of John Nakashima

http://www.areaguidebook.com/2005archives/Nakashima.htm



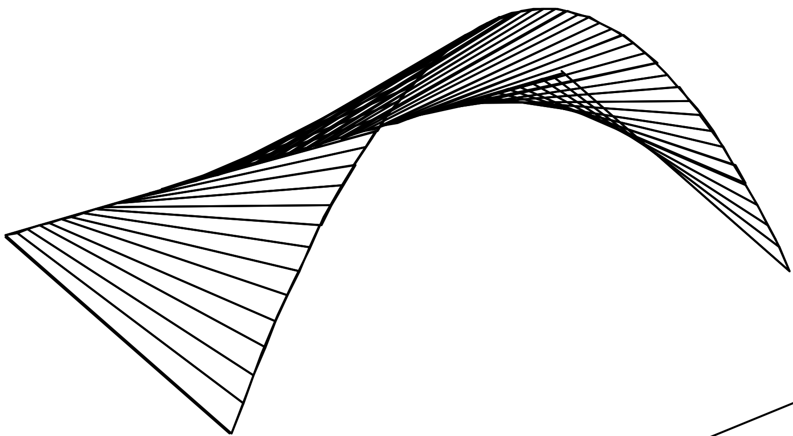
Sagrada Familia Parish School.

Despite it was merely a provisional building destined to be a school for the sons of the bricklayers working in the temple, it is regarded as one of the chief Gaudinian architectural works.

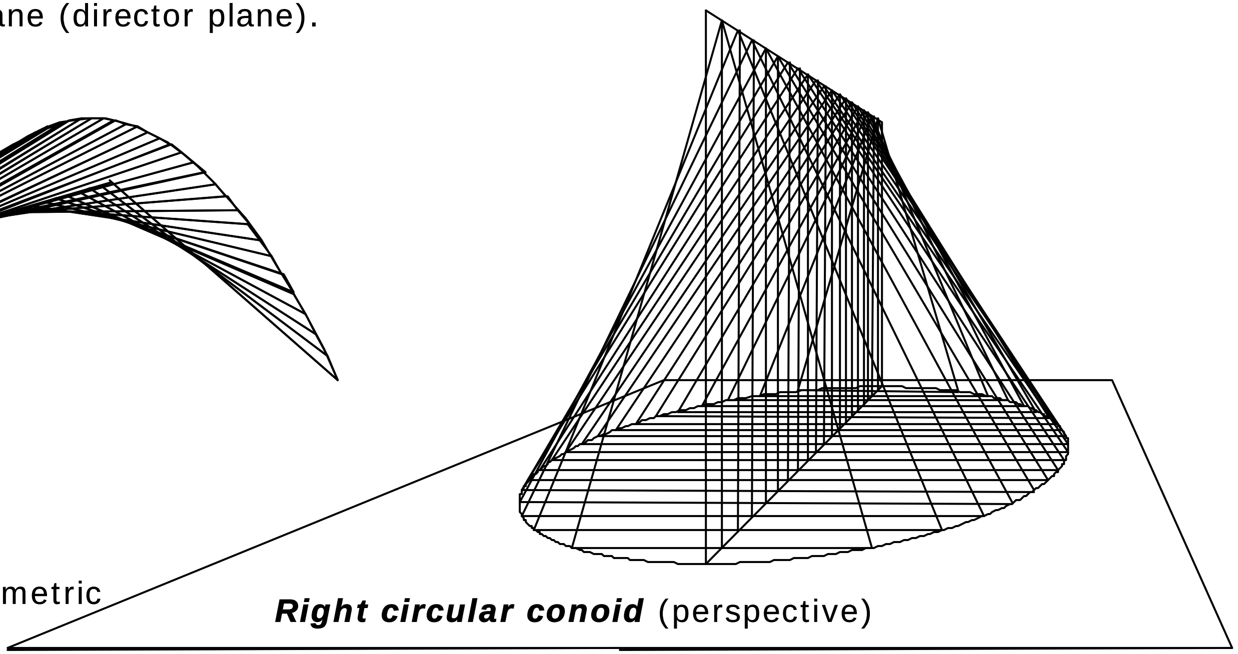
http://www.gaudiclub.com/ingles/i_VIDA/escoles.asp

Conoid

Definition: ruled surface, set of lines (rulings), which are transversals of a straight line (directrix) and a curve (base curve), parallel to a plane (director plane).



Parabola-conoid (axonometric sketch)



Right circular conoid (perspective)

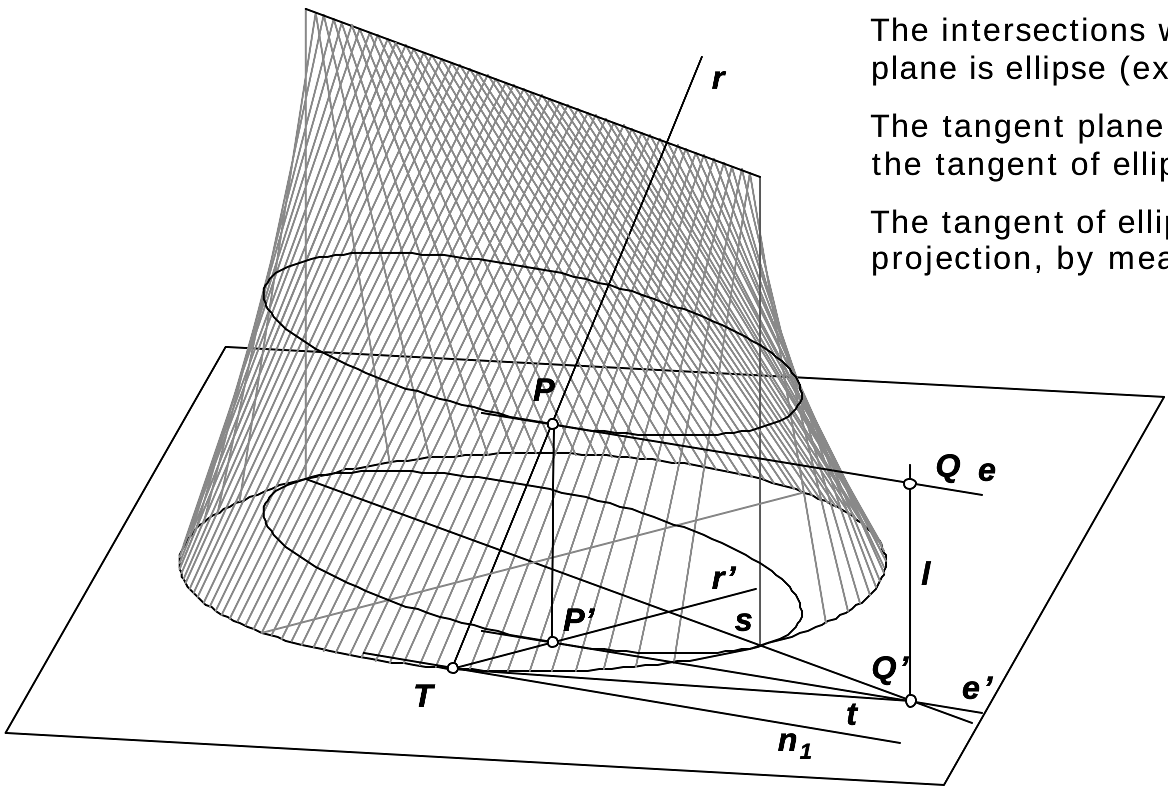
Eric W. Weisstein. "Right Conoid." From *MathWorld*--A Wolfram Web Resource.

http://mathworld.wolfram.com/RightConoid.html

http://mathworld.wolfram.com/PlueckersConoid.html

http://mathworld.wolfram.com/RuledSurface.html

Tangent Plane of the Right Circular Conoid at a Point

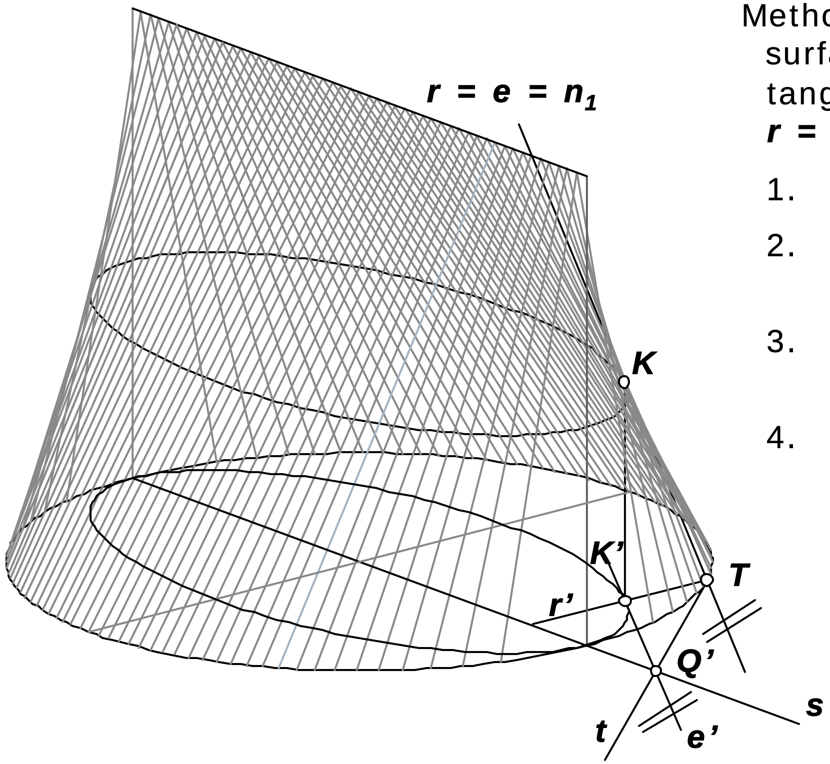


The intersections with a plane parallel to the base plane is ellipse (except the directrix).

The tangent plane is determined by the ruling and the tangent of ellipse passing through the point.

The tangent of ellipse is constructible in the projection, by means of affinity {***a***, ***P'***⇔ ***(P)***}

Contour of Conoid in Axonometry

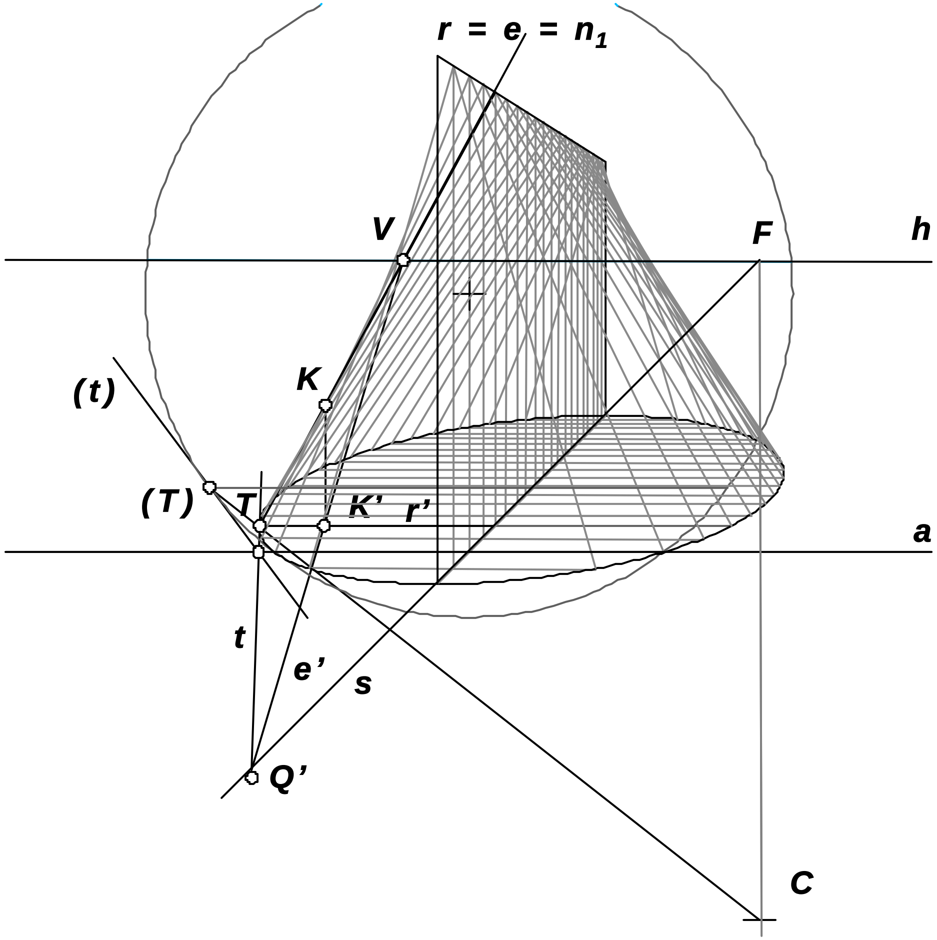


Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i. e. the ruling ***r***, the tangent of ellipse ***e*** and the tracing line n_1 coincide: $r = e = n_1$

1. Chose a ruling ***r***
2. Construct the tangent ***t*** of the base circle at the pedal point ***T*** of the ruling ***r***
3. Through the point of intersection of ***s*** and ***t***, ***Q'*** draw ***e'*** parallel to ***e***
4. The point of intersection of ***r'*** and ***e'***, ***K'*** is the projection of the contour point ***K***
5. Elevate the point ***K'*** to get ***K***

Contour of Conoid in Perspective



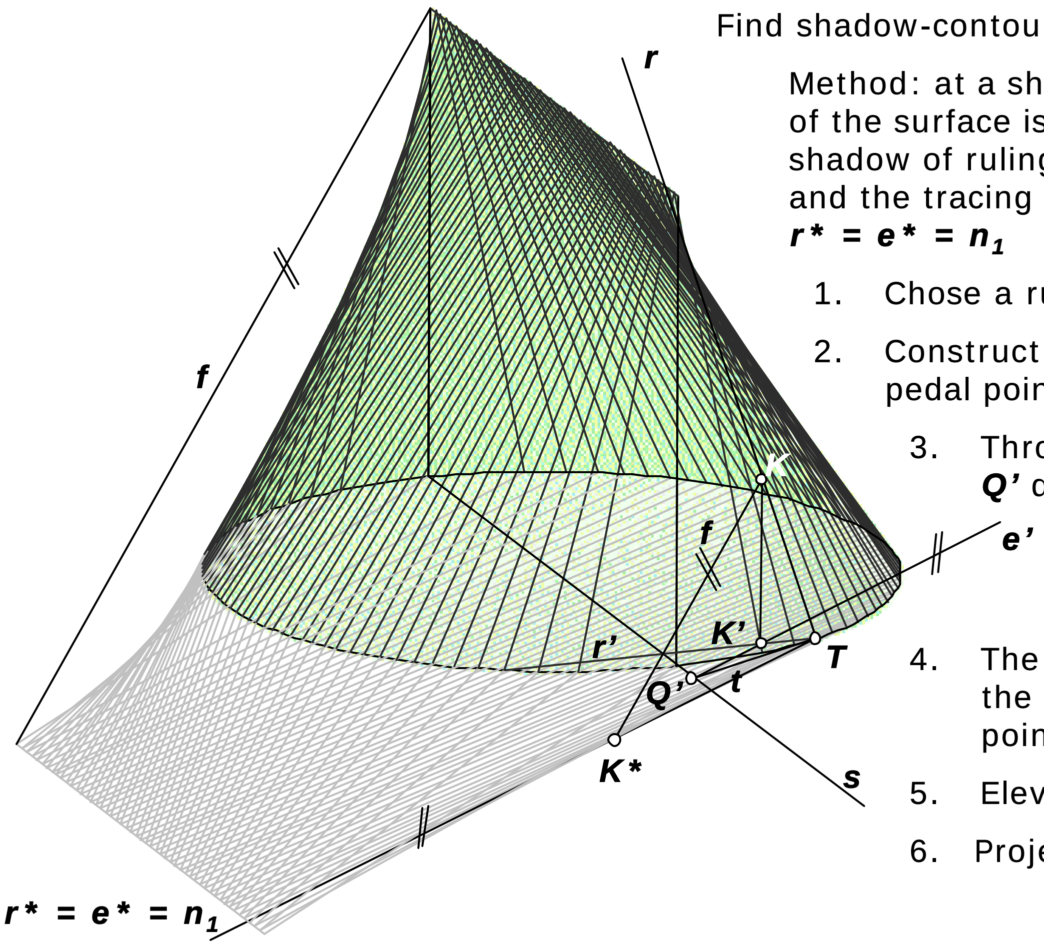
Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i. e. the ruling ***r***, the tangent of ellipse ***e*** and the tracing line n_1 coincide:

$r = e = n_1$

1. Chose a ruling ***r***
2. Construct the tangent ***t*** of the base circle at the pedal point ***T*** of the ruling ***r***
3. Through the point of intersection of ***s*** and ***t***, ***Q'*** draw ***e'*** parallel to ***e*** (***e*** ∩ ***e'*** = ***V*** -o- ***h***)
4. The point of intersection of ***r'*** and ***e'***, ***K'*** is the projection of the contour point ***K***
5. Elevate the point ***K'*** to get ***K***

Shadow of Conoid



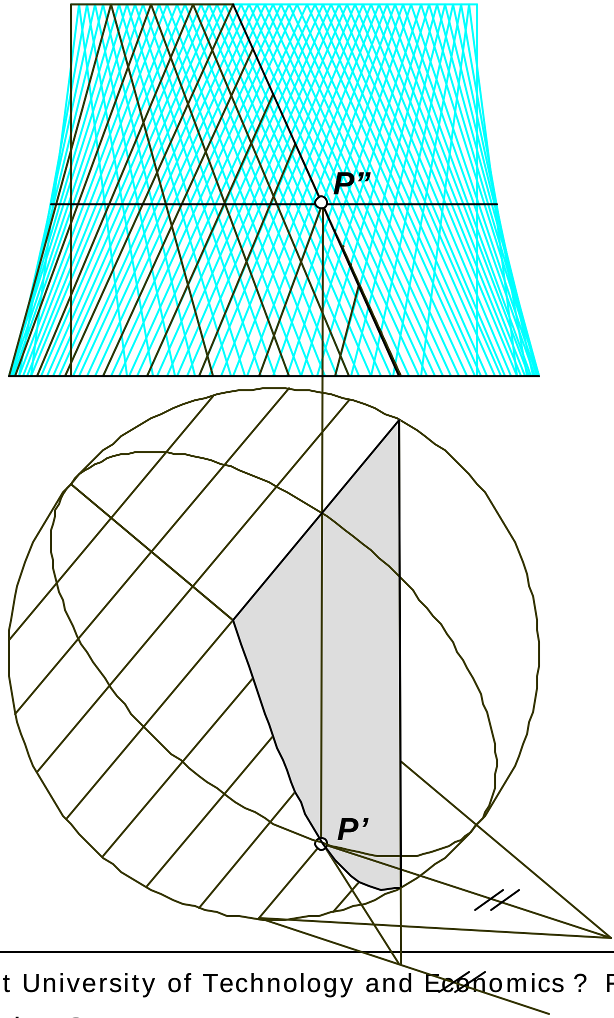
Find shadow-contour point of a ruling

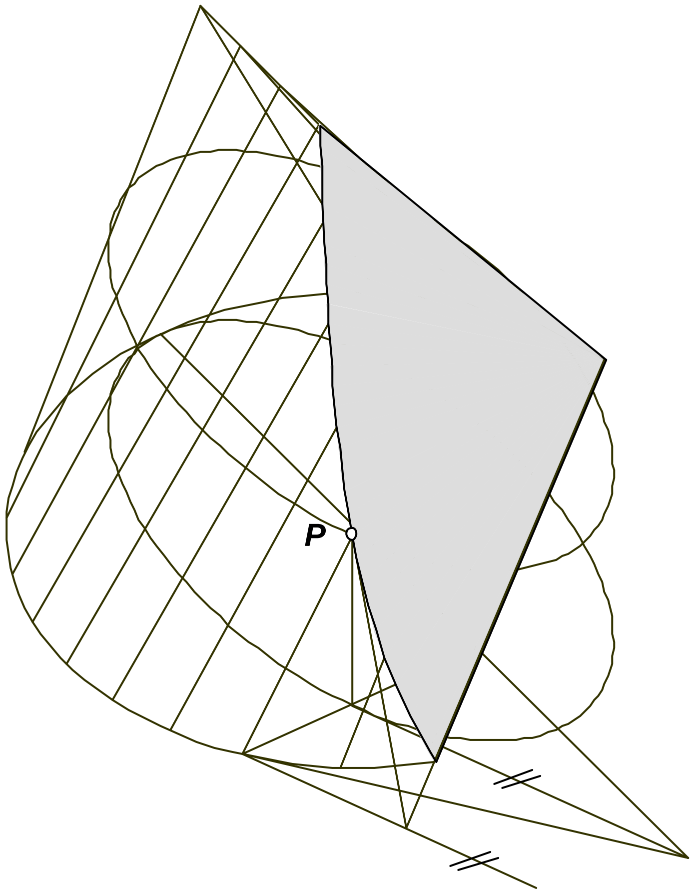
Method: at a shadow-contour point, the tangent plane of the surface is a shadow-projecting plane, i. e. the shadow of ruling ***r****, the shadow of tangent of ellipse ***e**** and the tracing line n_1 coincide:

$r^* = e^* = n_1$

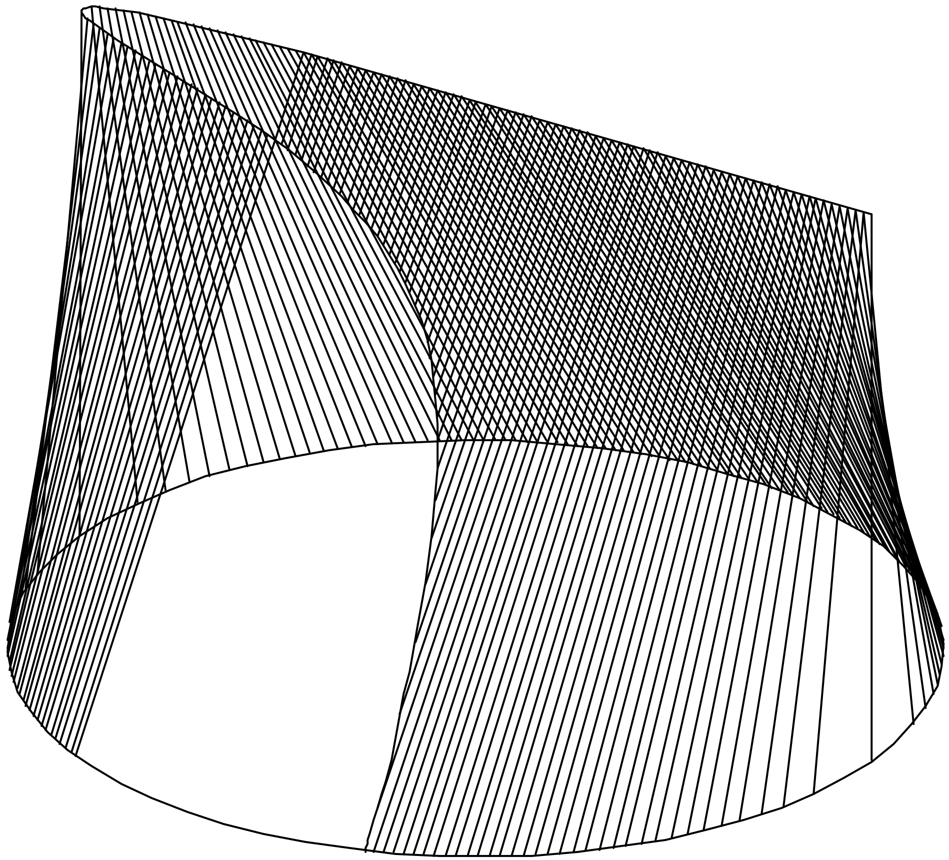
1. Chose a ruling ***r***
2. Construct the tangent ***t*** of the base circle at the pedal point ***T*** of the ruling ***r***
3. Through the point of intersection of ***s*** and ***t***, ***Q'*** draw ***e'*** parallel to ***e****
4. The point of intersection of ***r'*** and ***e'***, ***K'*** is the projection of the contour point ***K***, a point of the self-shadow outline
5. Elevate the point ***K'*** to get ***K***
6. Project ***K*** to get ***K****

Intersection of Conoid and Plane





Intersection of Conoid and Tangent Plane



Developable Surfaces

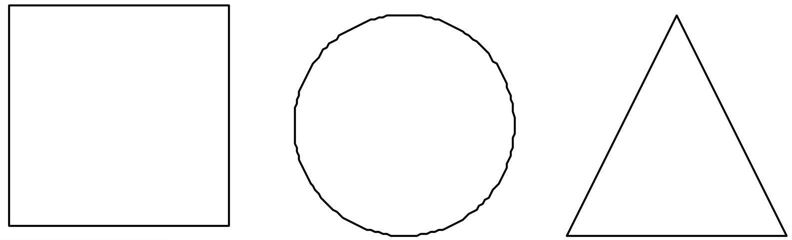
mARTa Herford - Frank Gehry ©

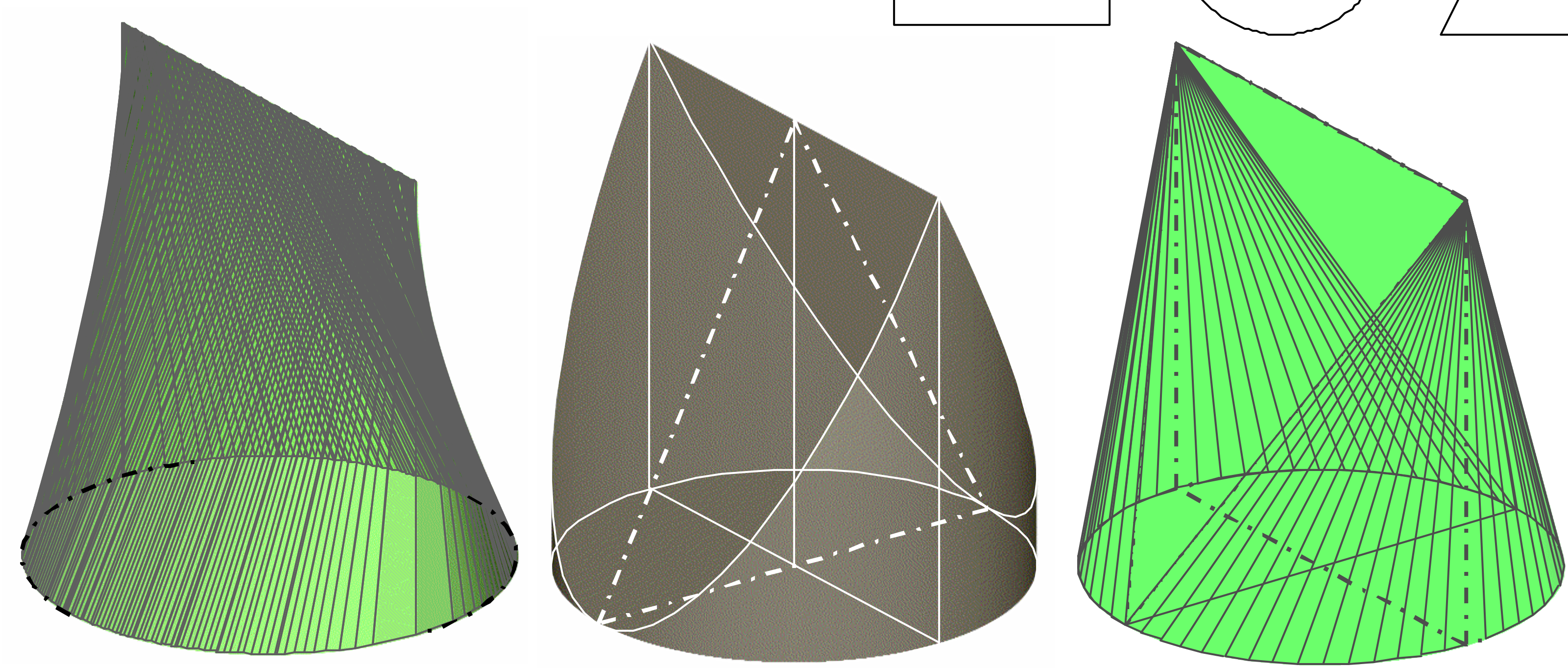
Developable surfaces can be unfolded onto the plane without stretching or tearing. This property makes them important for several applications in manufacturing.

http://www.geometrie.tuwien.ac.at/geom/bibtexing/devel.html
http://en.wikipedia.org/wiki/Developable_surface
http://www.rhino3.de/design/modeling/developable/
C:\Documents and Settings\Pali\Dokumentumok\palidok\Tematika\Tematikus\Developable\Rhino3DE Developable Surfaces.htm

Developable Surface

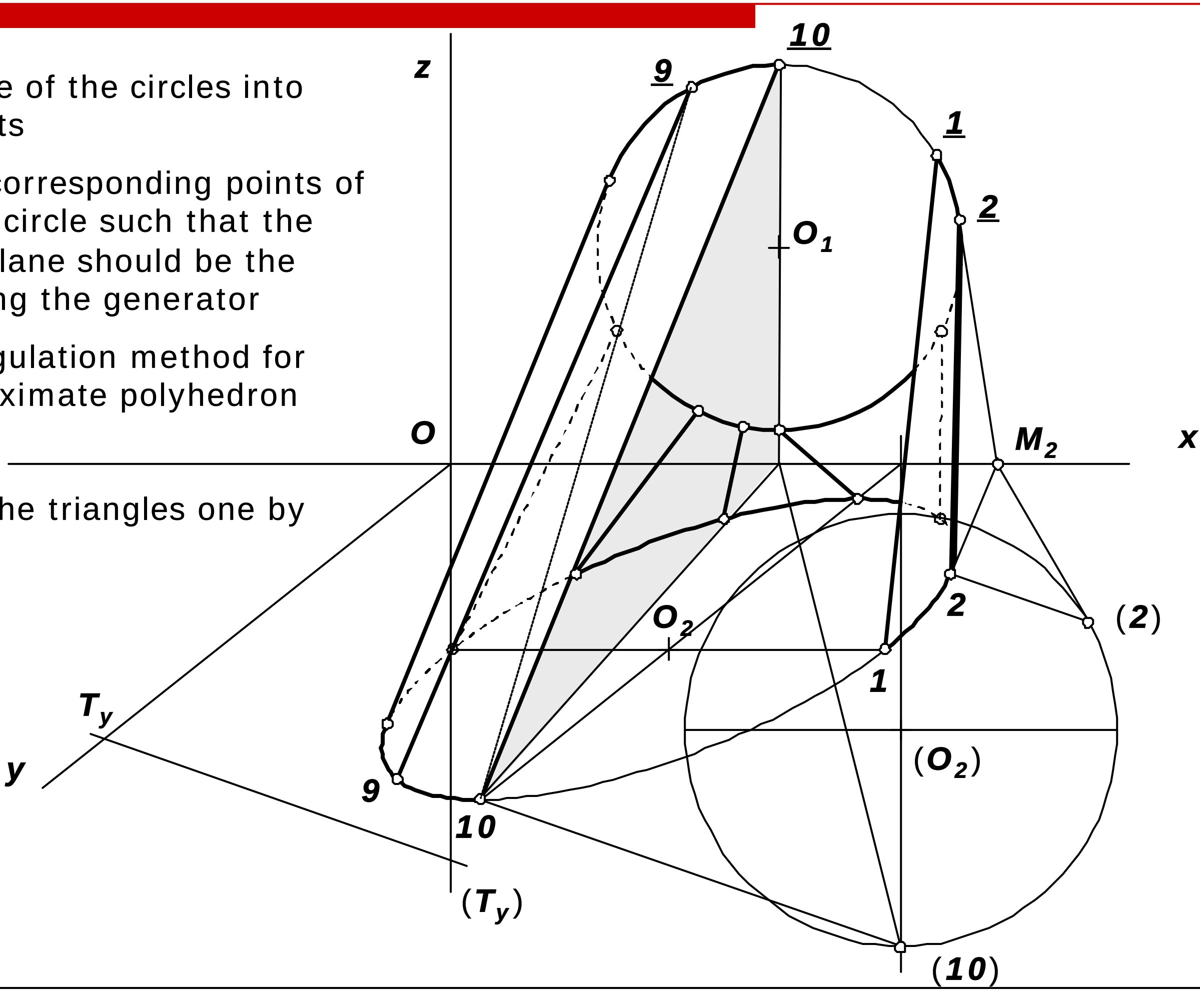
Find the proper plug that fits into the three plug-holes. The conoid is non-developable, the cylinder and the cone are developable.



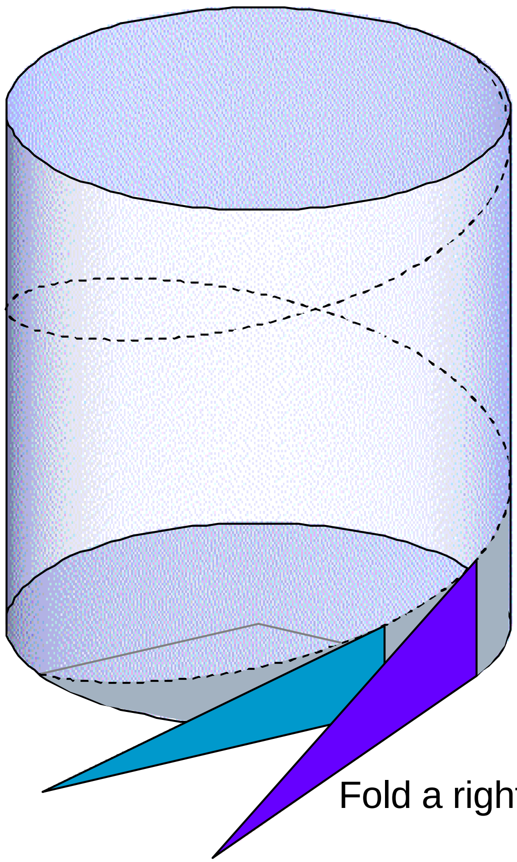


Developable Surface

1. Divide one of the circles into equal parts
2. Find the corresponding points of the other circle such that the tangent plane should be the same along the generator
3. Use triangulation method for the approximate polyhedron
4. Develop the triangles one by one

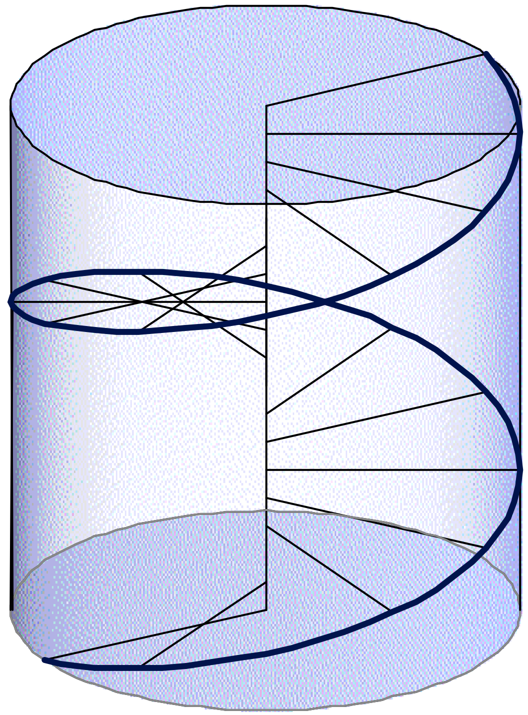


Helix



Fold a right triangle around a cylinder

Helical motion: rotation + translation



Left-handed, Right-handed Staircases

While elevating, the rotation about the axis is clockwise: left-handed

$$x(t) = a\sin(t)$$

$$y(t) = a\cos(t)$$

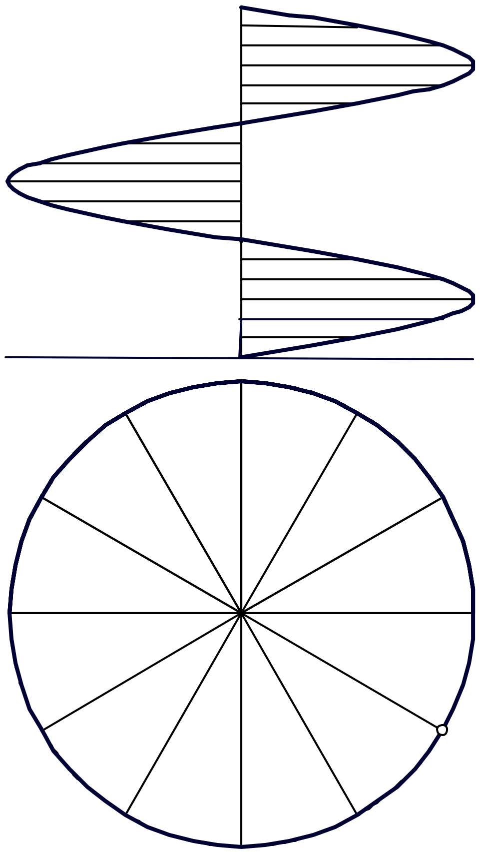
$$z(t) = c\,t$$

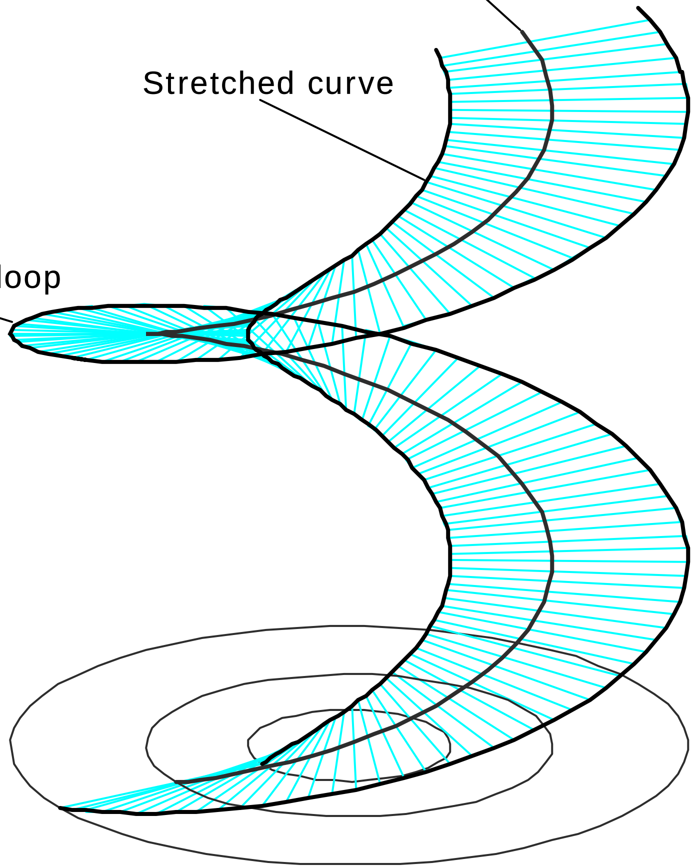
c > 0: right-handed

c < 0: left-hande

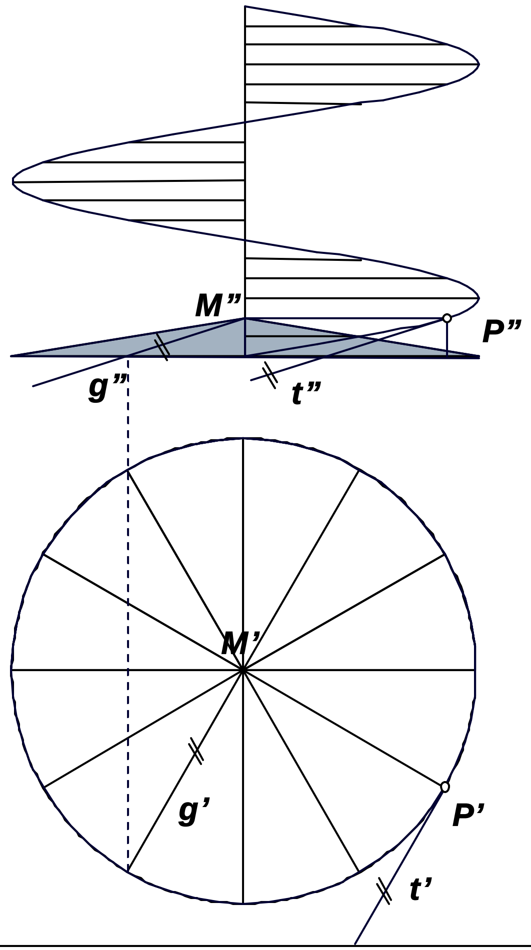
While elevating, the rotation about the axis is counterclockwise: right-handed

Classification of Images of Helix





Helix, Tangent, Director Cone



P: half of the perimeter

p: pitch

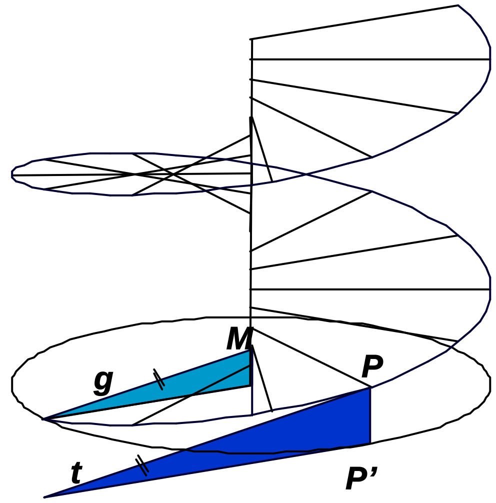
a: radius of the cylinder

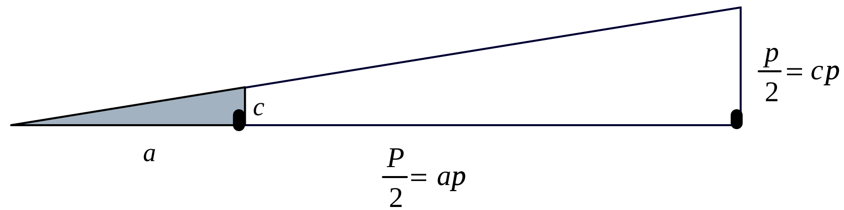
c: height of director cone = parameter of helical motion

M: vertex of director cone

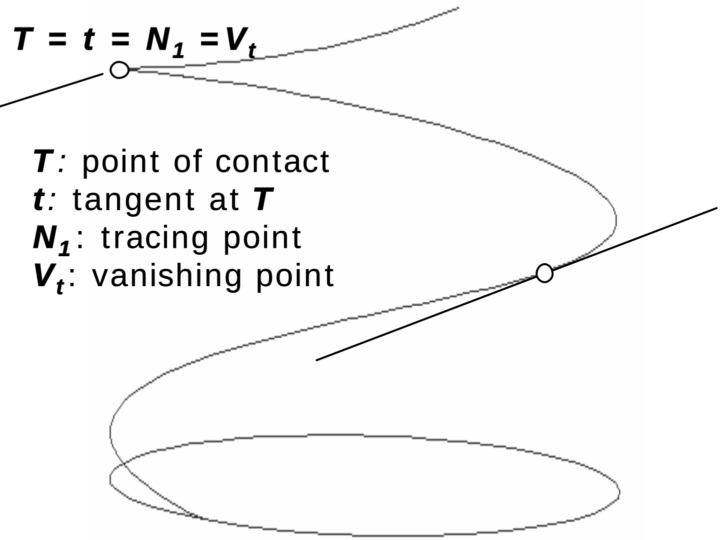
g: generator of director cone

t: tangent of helix





Helix with Cuspidal Point in Perspective

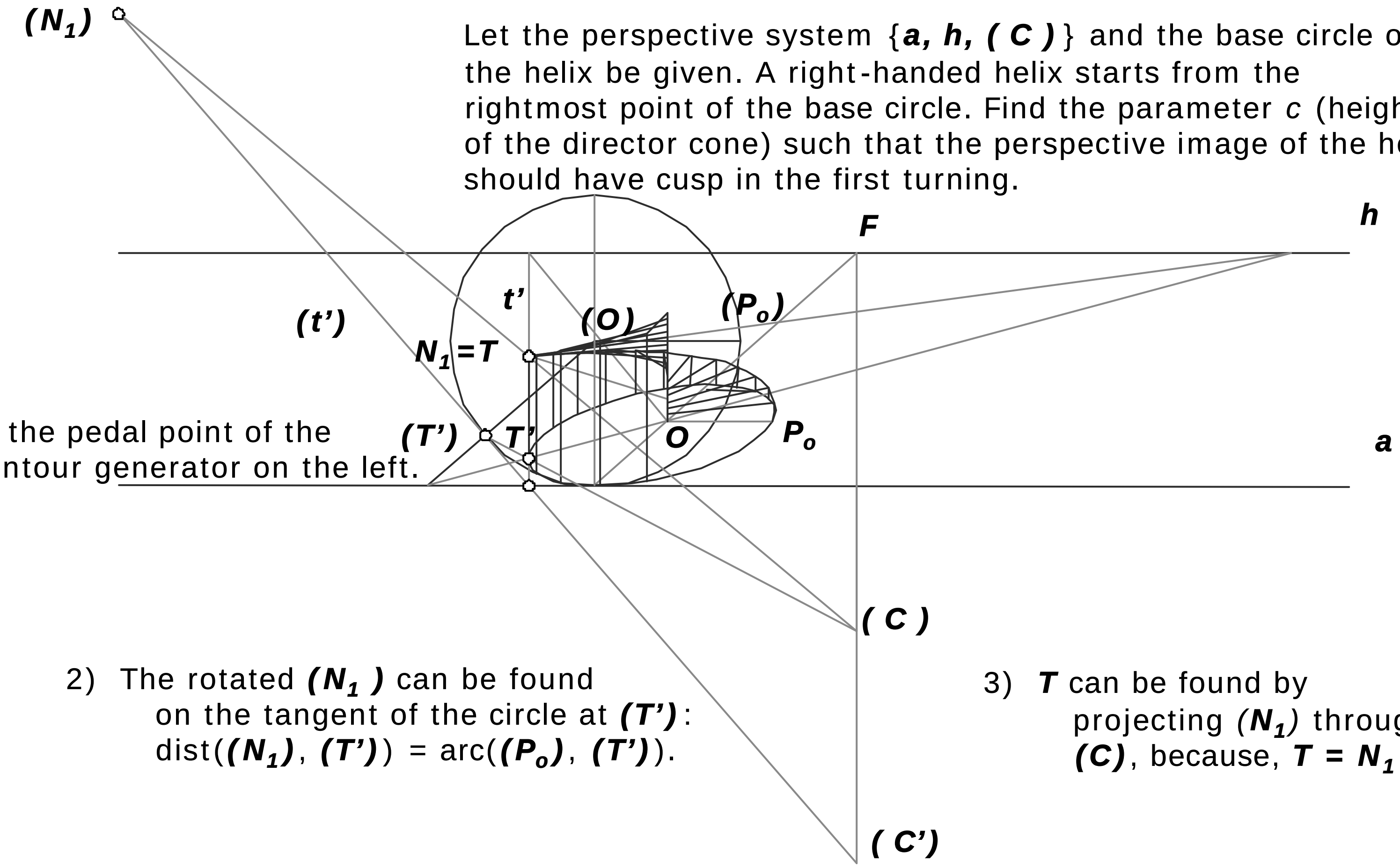


The tangent of the helix at cuspidal point is perspective projecting line: $T = t = N_1 = V_t$

Since t lies in a tangent plane of the cylinder of the helix, it lies on a contour generator of the cylinder (leftmost or rightmost)

Construction Helix with Cusp in Perspective 1

Let the perspective system {***a, h, (C)***} and the base circle of the helix be given. A right-handed helix starts from the rightmost point of the base circle. Find the parameter *c* (height of the director cone) such that the perspective image of the helix should have cusp in the first turning.



1) ***T'*** is the pedal point of the contour generator on the left.

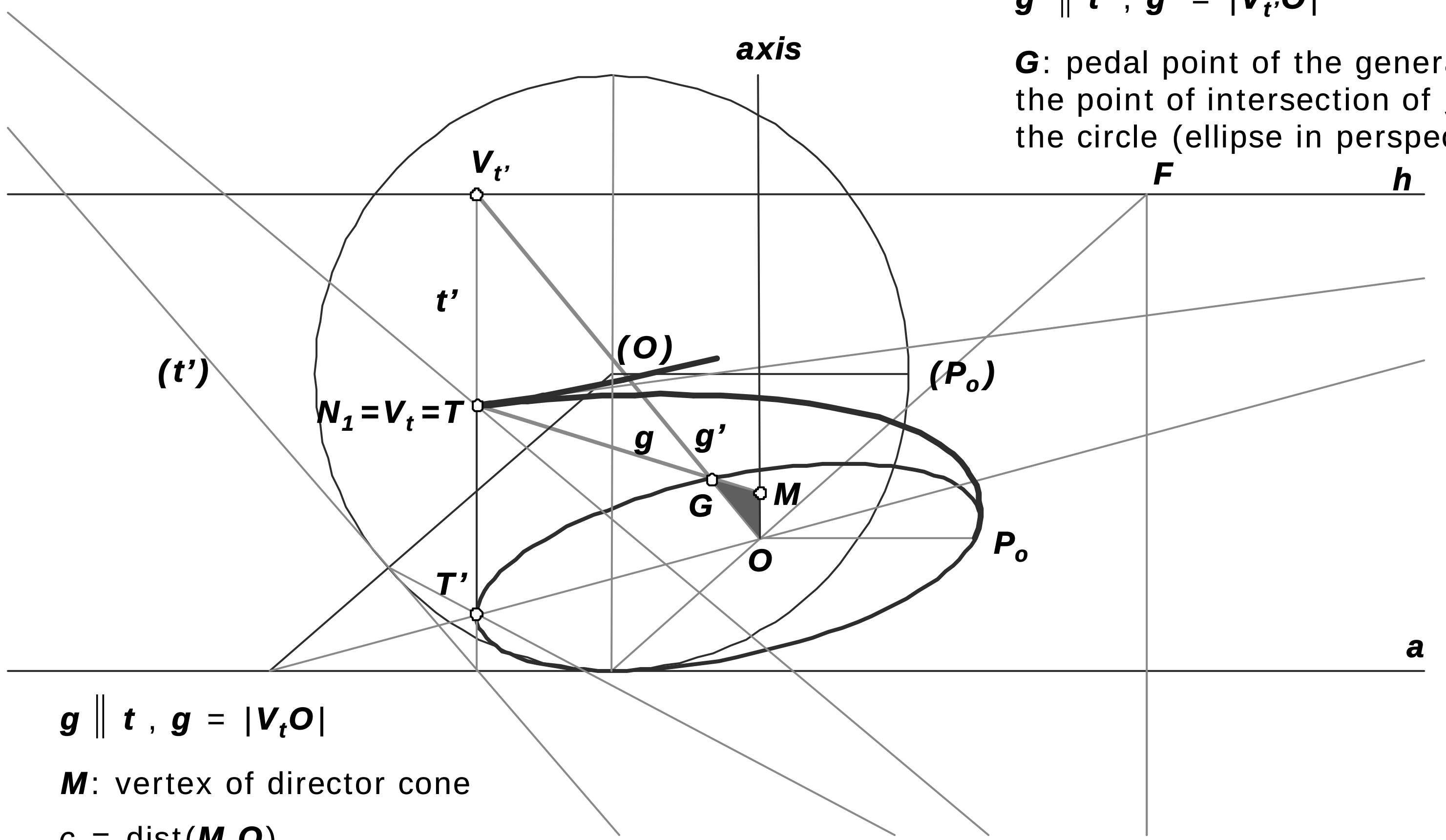
2) The rotated ***(N₁)*** can be found on the tangent of the circle at ***(T')***: dist((N_1), (T')) = arc((P_o), (T')).

3) ***T*** can be found by projecting (N_1) through ***(C)***, because, $T = N_1$.

Construction Helix with Cusp in Perspective 2

$g' \parallel t'$, $g' = |V_{t'}O|$

G: pedal point of the generator ***g***, the point of intersection of ***g'*** and the circle (ellipse in perspective)



$g \parallel t$, $g = |V_tO|$

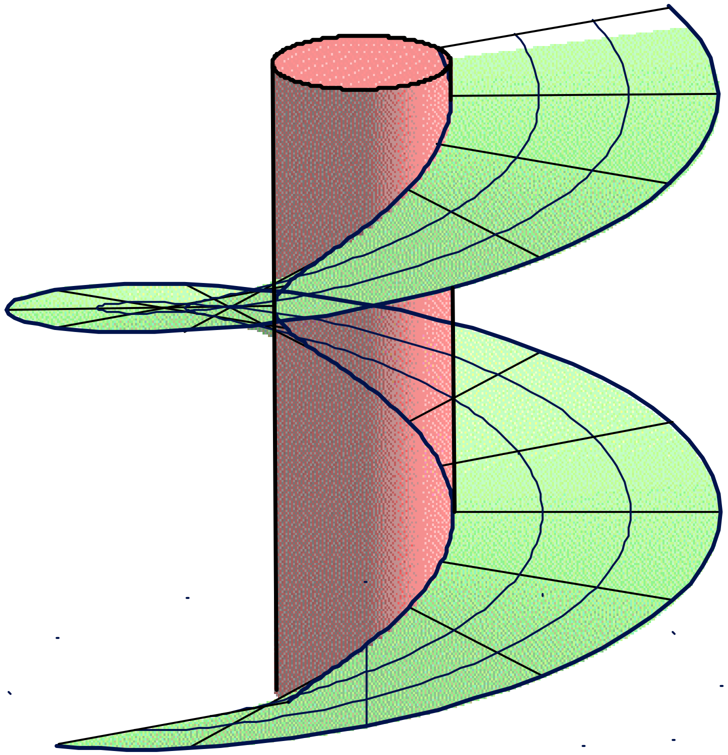
M: vertex of director cone

c = dist(M,O)

Helicoid

Definition:

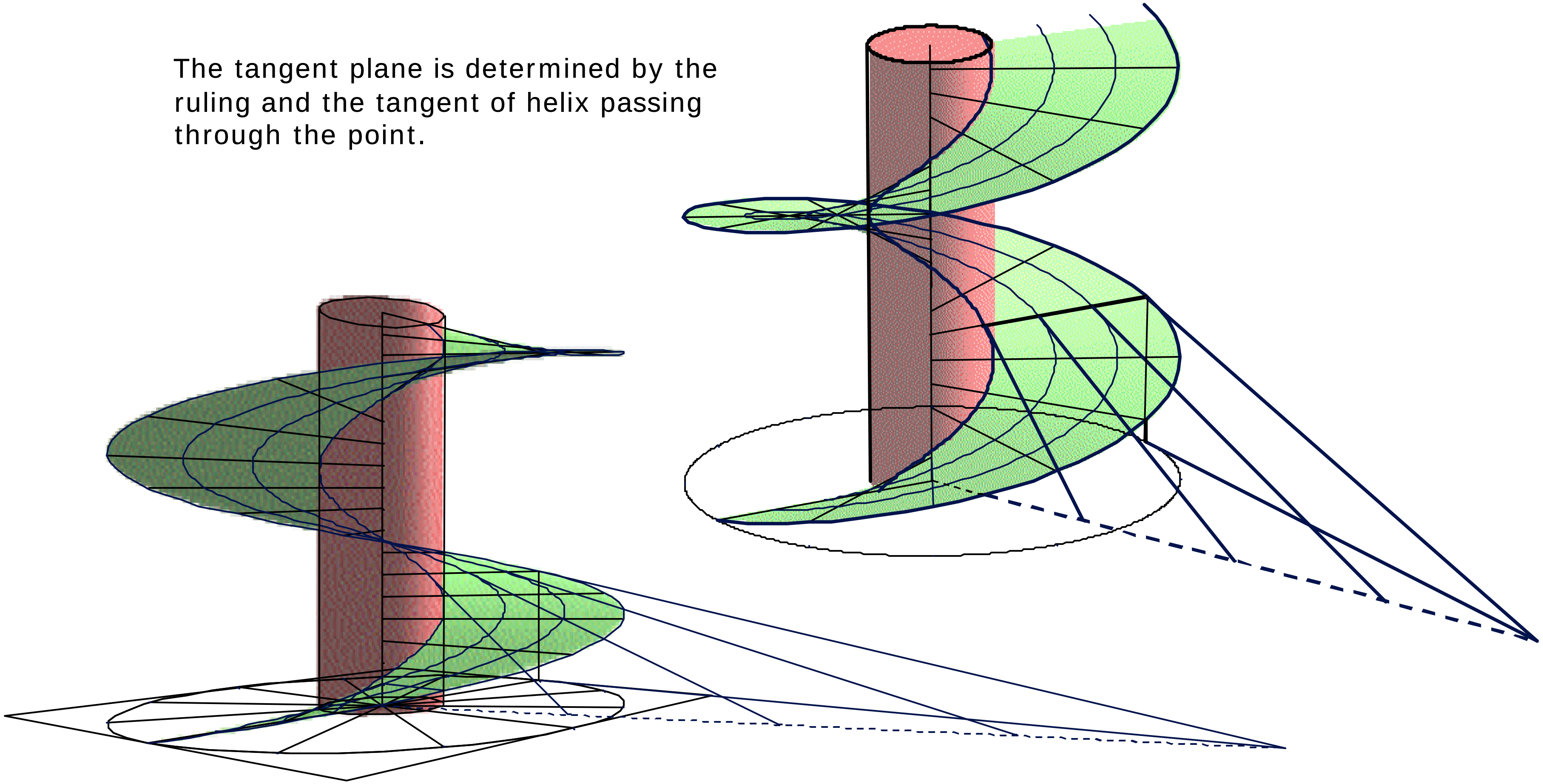
A ruled surface, which may be generated by a straight line moving such that every point of the line shall have a uniform motion in the direction of another fixed straight line (axis), and at the same time a uniform angular motion about it.



Eric W. Weisstein. "Helicoid." From *MathWorld*--A Wolfram Web Resource.
http://mathworld.wolfram.com/Helicoid.html
http://en.wikipedia.org/wiki/Helicoid
http://vmm.math.uci.edu/3D-XplorMath/Surface/helicoid-catenoid/helicoid-catenoid_lg1.html

Tangent Plane of Helicoid

The tangent plane is determined by the ruling and the tangent of helix passing through the point.

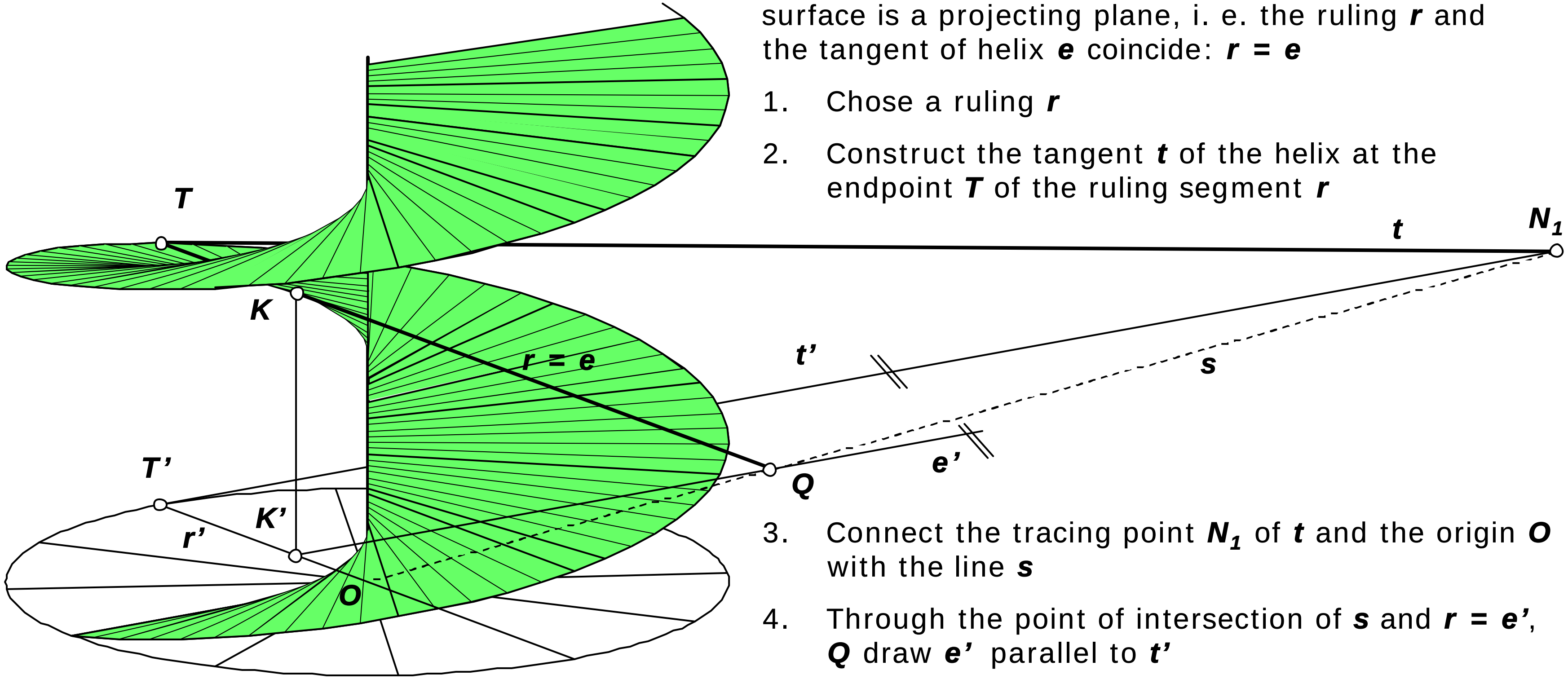


Contour of Helicoid

Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i. e. the ruling ***r*** and the tangent of helix ***e*** coincide: ***r = e***

1. Chose a ruling ***r***
2. Construct the tangent ***t*** of the helix at the endpoint ***T*** of the ruling segment ***r***



3. Connect the tracing point $\boldsymbol{N_1}$ of ***t*** and the origin ***O*** with the line ***s***
4. Through the point of intersection of ***s*** and ***r = e'***, ***Q*** draw ***e'*** parallel to ***t'***
5. The point of intersection of ***r'*** and ***e'***, ***K'*** is the projection of the contour point ***K***