Descriptive Geometry 2

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Conic Sections

Ellipse  Parabola  Hyperbola
Intersection of Cone and Plane: Ellipse

The intersection of a cone of revolution and a plane is an ellipse if the plane (not passing through the vertex of the cone) intersects all generators.

Dandelin spheres: spheres in a cone, tangent to the cone (along a circle) and also tangent to the plane of intersection.

Foci of ellipse: $F_1$ and $F_2$, points of contact of plane of intersection and the Dandelin spheres.

$P$: piercing point of a generator, point of the curve of intersection.

$T_1$ and $T_2$, points of contact of the generator and the Dandelin sphere.

$PF_1 = PT_1$, $PF_2 = PT_2$ (tangents to sphere from an external point).

$PF_1 + PF_2 = PT_1 + PT_2 = T_1T_2 = \text{constant}$

http://www.clowder.net/hop/Dandelin/Dandelin.html
Construction of Minor Axis

Let the plane of intersection $\alpha''$ second projecting plane that intersects all generators.

The endpoints of the major axis are $A$ and $B$, the piercing points of the leftmost and rightmost generators respectively.

The midpoint $L$ of $AB$ is the centre of ellipse.

Horizontal auxiliary plane $\beta''$ passing through $L$ intersects the cone in a circle with the centre of $K$. ($K$ is a point of the axis of the cone).

The endpoints of the minor axis $C$ and $D$ can be found as the points of intersection of the circle in $\beta$ and the reference line passing through $L''$. 
Intersection of Cone and Plane: Parabola

The intersection of a cone of revolution and a plane is a parabola if the plane (not passing through the vertex of the cone) is parallel to one generator.

Focus of parabola: $F$, point of contact of the plane of intersection and the Dandelin sphere.

Directrix of parabola: $d$, line of intersection of the plane of intersection and the plane of the circle on the Dandelin sphere.

$P$: piercing point of a generator, point of the curve of intersection.

$T$: point of contact of the generator and the Dandelin sphere.

$PF = PT$ (tangents to sphere from an external point).

$PT = T_1T_2 = PE$, $dist(P,F) = dist(P,d)$. 

http://mathworld.wolfram.com/DandelinSpheres.html
Construction of Point and Tangent

Let the plane of intersection $\alpha^\prime\prime$ second projecting plane be parallel to the rightmost generator.

The vertex of the parabola is $V$.

Horizontal auxiliary plane $\beta^\prime\prime$ can be used to find $P^\prime\prime$, the second image of a point of the parabola.

The tangent $t$ at a point $P$ is the line of intersection of the plane of intersection and the tangent plane of the surface at $P$.

The first tracing point of the tangent $N_1$ is the point of intersection of the first tracing line of the plane of intersection and the first tracing line of the tangent plane at $P$, $n_{11}$ and $n_{12}$ respectively.

$$t = |N_1P|$$
Intersection of Cone and Plane: Hyperbola

The intersection of a cone of revolution and a plane is a *hyperbola* if the plane (not passing through the vertex of the cone) is parallel to two generators.

Foci of hyperbola: $F_1$ and $F_2$, points of contact of plane of intersection and the Dadelin spheres.

$P$: piercing point of a generator, point of the curve of intersection.

$T_1$ and $T_2$, points of contact of the generator and the Dandelin spheres.

$PF_1 = PT_1$, $PF_2 = PT_2$ (tangents to sphere from an external point).

$PF_2 - PF_1 = PT_2 - PT_1 = T_1T_2 = \text{constant.}$
Construction of Asymptotes

Let the plane of intersection $\alpha$" second projecting plane parallel to two generators, that means, parallel to the second projecting plane $\beta$ through the vertex of the cone, which intersects the cone in two generators $g_1$ and $g_2$.

The endpoints of the traverse (real) axis are $A$ and $B$, the piercing points of the two extreme generators.

The midpoint $L$ of $AB$ is the centre of hyperbola.

The asymptotic lines $a_1$ and $a_2$ are the lines of intersections of the tangent planes along the generators $g_1$ and $g_2$ and the plane of intersection $\alpha$. 
Perspective Image of Circle

- Horizon
- Axis
- Vanishing line
Construction of Perspective Image of Circle

The distance of the horizon and the \((C)\) is equal to the distance of the axis and the vanishing line.

The type of the perspective image of a circle depends on the number of common points with the vanishing line:

- no point in common; ellipse
- one point in common; parabola
- two points in common; hyperbola

The asymptotes of the hyperbola are the projections of the tangents at the vanishing points.
Tangent Planes, Surface Normals
Intersection of Cone and Cylinder 1
Intersection of Cone and Cylinder 2
Intersection of Cone and Cylinder 3
Methods for Construction of Point 1

Auxiliary plane:

plane passing through the vertex of the cone and parallel to the axis of cylinder
Methods for Construction of Point 2

Auxiliary plane:
first principal plane
Principal Points

Double point: \( D \)

Points in the plane of symmetry: \( S_1, S_2 \)

Points On the outline of cylinder: \( K_1, K_2, K_3, K_4 \)

Points On the outline of cone: \( K_5, K_6, K_7, K_8 \)
Surfaces of Revolution: Ellipsoid

Prolate ellipsoid

Oblate ellipsoid

Axis plane of affinity

Affinity

Capitol
Ellipsoid of Revolution in Orthogonal Axonometry
Intersection of Ellipsoid and Plane

http://www.burgstaller-arch.at/
Surfaces of Revolution: Paraboloid
Paraboloid; Shadows

Find
- the focus of parabola
- tangent parallel to f”
- self-shadow
- cast shadow
- projected shadow inside
Torus

- **equatorial circle**
- **throat circle**
- **meridian circle**
- **axis of rotation** (z)

Diagram labels:
- z
- y
- x
**Classification of Toruses**

<table>
<thead>
<tr>
<th>full view</th>
<th>cutaway</th>
<th>cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ring torus</strong></td>
<td><img src="image1" alt="Ring Torus" /></td>
<td><img src="image2" alt="Ring Torus Cross-Section" /></td>
</tr>
<tr>
<td><strong>horn torus</strong></td>
<td><img src="image3" alt="Horn Torus" /></td>
<td><img src="image4" alt="Horn Torus Cross-Section" /></td>
</tr>
<tr>
<td><strong>spindle torus</strong></td>
<td><img src="image5" alt="Spindle Torus" /></td>
<td><img src="image6" alt="Spindle Torus Cross-Section" /></td>
</tr>
</tbody>
</table>

Torus as Envelope of Spheres
Outline of Torus as Envelope of Circles
Outline of Torus
Classification of Points of Surface

- **hyperbolical** (yellow)
- **elliptical** (blue)
- **parabolical** (two circles)
Tangent plane at Hyperbolic Point
Villarceau Circles
Construction of Contour and Shadow
Ruled Surface

http://www.cs.mtu.edu/~shene/COURSES/cs3621/LAB/surface/ruled.html
http://mathworld.wolfram.com/RuledSurface.html
http://en.wikipedia.org/wiki/Ruled_surface
http://www.geometrie.tuwien.ac.at/havlicek/torse.html
http://www.f.waseda.jp/takezawa/mathenglish/geometry/surface2.htm
http://www.amsta.leeds.ac.uk/~khouston/ruled.htm
Hyperboloid of One Sheet

http://www.archinform.net/mediaw/00004570.htm?ID=25c5e478c9a503703d984a4e4803ded

http://www.jug.net/wt/slscp/slscpa.htm

http://www.visualjourney.com

Hyperboloid of One Sheet
Hyperboloid of One Sheet, Surface of Revolution
Shadows on Hyperboloid of one Sheet
Hyperboloid of One Sheet, Shadow 1

The self-shadow outline is the hyperbola $h$ in the plane of the self-shadow generators of asymptotic cone.

The cast-shadow on the hyperboloid itself is an arc of ellipse $e$ inside of the surface.
Hyperboloid of One Sheet, Shadow 2

The self-shadow outline is the ellipse $e_1$ inside of the surface.

The cast-shadow on the hyperboloid itself is an arc of ellipse $e_2$ on the outer side of the surface.
The outline of the self-shadow is a pair of parallel generators \( g_1 \) and \( g_2 \).
Hyperboloid of One Sheet in Perspective
Hyperboloid in Military Axonometry
Construction of Self-shadow and Cast Shadow

$V^*$: shadow of the center $V$ that is the vertex of the asymptotic cone

t: tangent to the throat circle, chord of the base circle

d: parallel and equal to $t$ through the center of the base circle, diameter of the asymptotic cone

Tangents to the base circle of the asymptotic cone from $V^*$ with the points of contact $A_1$ and $A_2$: asymptotes of the cast-shadow outline hyperbola

Line $A_1A_2$: first tracing line of the plane of self-shadow hyperbola; $VA_1$ and $VA_2$: asymptotes of the self-shadow outline hyperbola
Construction of Projected Shadow

The cast-shadow of $H$ on the ground plane: $H^*$

Draw a generator $g_1$ passing through $H$

Find $B_1$, the pedal point of the generator $g_1$

Find $n_1 = B_1H^*$ first tracing line of the auxiliary plane $HB_1H^*$

The point of intersection of the base circle and $n_1$ is $B_2$

The second generator $g_2$ lying in the plane $HB_1H^*$ intersects the ray of light $l$ at $H$, the lowest point of the ellipse, i.e. the outline of the projected shadow inside.
Hyperboloid of One Sheet with Horizontal Axis
Hyperboloid of One Sheet, Intersection with Sphere
Ruled Surface: Hyperbolic Paraboloid

http://www.recentpast.org/types/hyperpara/index.html

http://www.ketchum.org/shellpix.html#airform
Hyperbolic Paraboloid: Construction

http://www.anangpur.com/struc7.html
Saddle Surface

http://emsh.calarts.edu/~mathart/Annotated_HyperPara.html
Axonometry and Perspective
Saddle Point an Contour
Shadow at Parallel Lighting
Intersection with Cylinder
Composite Surface
Intersection with Plane
Conoid

Conoid Studio, Interior.
Photo by Ezra Stoller (c)ESTO
Courtesy of John Nakashima


Sagrada Familia Parish School.
Despite it was merely a provisional building destined to be a school for the sons of the bricklayers working in the temple, it is regarded as one of the chief Gaudinian architectural works.

http://www.gaudiclub.com/ingles/i_VIDA/escoles.asp
Conoid

**Definition:** ruled surface, set of lines (rulings), which are transversals of a straight line (directrix) and a curve (base curve), parallel to a plane (director plane).

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**Parabola-conoid** (axonometric sketch)

**Right circular conoid** (perspective)


http://mathworld.wolfram.com/RightConoid.html
http://mathworld.wolfram.com/PlueckersConoid.html
http://mathworld.wolfram.com/RuledSurface.html
Tangent Plane of the Right Circular Conoid at a Point

The intersections with a plane parallel to the base plane is ellipse (except the directrix).

The tangent plane is determined by the ruling and the tangent of ellipse passing through the point.

The tangent of ellipse is constructible in the projection, by means of affinity \( \{a, P \rightarrow (P)\} \).
Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i.e. the ruling \( r \), the tangent of ellipse \( e \) and the tracing line \( n_1 \) coincide: \( r = e = n_1 \)

1. Chose a ruling \( r \)
2. Construct the tangent \( t \) of the base circle at the pedal point \( T \) of the ruling \( r \)
3. Through the point of intersection of \( s \) and \( t \), \( Q' \) draw \( e' \) parallel to \( e \)
4. The point of intersection of \( r' \) and \( e' \), \( K' \) is the projection of the contour point \( K \)
5. Elevate the point \( K' \) to get \( K \)
Contour of Conoid in Perspective

Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i.e. the ruling $r$, the tangent of ellipse $e$ and the tracing line $n_1$ coincide:

$$r = e = n_1$$

1. Chose a ruling $r$
2. Construct the tangent $t$ of the base circle at the pedal point $T$ of the ruling $r$
3. Through the point of intersection of $s$ and $t$, $Q'$ draw $e'$ parallel to $e$ ($e \cap e' = V \rightarrow h$)
4. The point of intersection of $r'$ and $e'$, $K'$ is the projection of the contour point $K$
5. Elevate the point $K'$ to get $K$
Shadow of Conoid

Find shadow-contour point of a ruling

Method: at a shadow-contour point, the tangent plane of the surface is a shadow-projecting plane, i.e. the shadow of ruling \( r^* \), the shadow of tangent of ellipse \( e^* \) and the tracing line \( n_1 \) coincide:
\[ r^* = e^* = n_1 \]

1. Chose a ruling \( r \)
2. Construct the tangent \( t \) of the base circle at the pedal point \( T \) of the ruling \( r \)
3. Through the point of intersection of \( s \) and \( t \), \( Q' \) draw \( e' \) parallel to \( e^* \)
4. The point of intersection of \( r' \) and \( e' \), \( K' \) is the projection of the contour point \( K \), a point of the self-shadow outline
5. Elevate the point \( K' \) to get \( K \)
6. Project \( K \) to get \( K^* \)
Intersection of Conoid and Plane
Intersection of Conoid and Tangent Plane
Developable Surfaces

Developable surfaces can be unfolded onto the plane without stretching or tearing. This property makes them important for several applications in manufacturing.

mARTa Herford - Frank Gehry ©

http://www.geometrie.tuwien.ac.at/geom/bibtexing/devel.html
http://www.rhino3.de/design/modeling/developable/
C:\Documents and Settings\Pali\Dokumentumok\palidok\Tematika\Tematikus\Developable\Rhino3DE Developable Surfaces.htm
Developable Surface

Find the proper plug that fits into the three plug-holes. The conoid is non-developable, the cylinder and the cone are developable.
Developable Surface

1. Divide one of the circles into equal parts
2. Find the corresponding points of the other circle such that the tangent plane should be the same along the generator
3. Use triangulation method for the approximate polyhedron
4. Develop the triangles one by one
Helix

Fold a right triangle around a cylinder

Helical motion: rotation + translation
Left-handed, Right-handed Staircases

While elevating, the rotation about the axis is clockwise: left-handed

\[ x(t) = a \sin(t) \]
\[ y(t) = a \cos(t) \]
\[ z(t) = c \ t \]

\[ c > 0: \text{right-handed} \]
\[ c < 0: \text{left-handed} \]

While elevating, the rotation about the axis is counterclockwise: right-handed
Classification of Images of Helix

Sine curve, circle

Curve with cusp

Stretched curve

Curve with loop
Helix, Tangent, Director Cone

\( P \): half of the perimeter

\( p \): pitch

\( a \): radius of the cylinder

\( c \): height of director cone = parameter of helical motion

\( M \): vertex of director cone

\( g \): generator of director cone

\( t \): tangent of helix

\[
\frac{P}{2} = c\pi
\]

\[
\frac{P}{2} = a\pi
\]
Helix with Cuspidal Point in Perspective

The tangent of the helix at cuspidal point is perspective projecting line:

\[ T = t = N_1 = V_t \]

Since \( t \) lies in a tangent plane of the cylinder of the helix, it lies on a contour generator of the cylinder (leftmost or rightmost)
Construction Helix with Cusp in Perspective 1

Let the perspective system \( \{a, h, (C)\} \) and the base circle of the helix be given. A right-handed helix starts from the rightmost point of the base circle. Find the parameter \( c \) (height of the director cone) such that the perspective image of the helix should have cusp in the first turning.

1) \( T' \) is the pedal point of the contour generator on the left.

2) The rotated \( (N_1) \) can be found on the tangent of the circle at \( (T') \):

\[
\text{dist}((N_1), (T')) = \text{arc}((P_0), (T')).
\]

3) \( T \) can be found by projecting \( (N_1) \) through \( (C) \), because, \( T = N_1 \).
Construction Helix with Cusp in Perspective 2

\[ g' \parallel t', \quad g' = |V_tO| \]

\( G \): pedal point of the generator \( g \), the point of intersection of \( g' \) and the circle (ellipse in perspective)

\( M \): vertex of director cone

\( c = \text{dist}(M,O) \)
**Helicoid**

**Definition:**

A ruled surface, which may be generated by a straight line moving such that every point of the line shall have a uniform motion in the direction of another fixed straight line (axis), and at the same time a uniform angular motion about it.

http://mathworld.wolfram.com/Helicoid.html
http://en.wikipedia.org/wiki/Helicoid
Tangent Plane of Helicoid

The tangent plane is determined by the ruling and the tangent of helix passing through the point.
Contour of Helicoid

Find contour point of a ruling

Method: at a contour point, the tangent plane of the surface is a projecting plane, i.e., the ruling \( r \) and the tangent of helix \( e \) coincide: \( r = e \)

1. Chose a ruling \( r \)
2. Construct the tangent \( t \) of the helix at the endpoint \( T \) of the ruling segment \( r \)
3. Connect the tracing point \( N_1 \) of \( t \) and the origin \( O \) with the line \( s \)
4. Through the point of intersection of \( s \) and \( r = e' \), \( Q \) draw \( e' \) parallel to \( t' \)
5. The point of intersection of \( r' \) and \( e' \), \( K' \) is the projection of the contour point \( K \)