Descriptive Geometry 1

by Pál Ledneczki Ph.D.

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Introduction

About the purposes of studying Descriptive Geometry:

1. *Methods* and *“means”* for solving 3D geometrical construction problems. In this sense *Descriptive Geometry* is a branch of Geometry.

2. 2D representation of 3D technical object, i.e. basics of Technical Drawing, “*instrument*” in technical communication.

What is Descriptive Geometry?

„One simply takes two planes at right angles to each other, one vertical and the other horizontal then projects the figure to be represented orthogonally on these planes, the projections of all edges and vertices being clearly indicated. The projection on the vertical plane is known as the „elevation”, the other projection is called „the plan”. Finally, the vertical plane is folded about the line of intersection of the two planes until it also is horizontal. This puts on one flat sheet of paper what we ordinarily visualize in 3D”.

(A History of Mathematics by Carl B. Boyer, John Wiley & Sons, New York, 1991)

Gaspard Monge (1746 — 1818) was sworn not to divulge the above method and for 15 years, it was a jealously guarded military secret. Only in 1794, he was allowed to teach it in public at the Ecole Normale, Paris where Lagrange was among the auditors. „With his application of analysis to geometry, this devil of a man will make himself immortal”, exclaimed Lagrange.

R.Parthasarathy

http://en.wikipedia.org/wiki/Gaspard_Monge

About Descriptive Geometry 1

Methodology

- Multi-view representation, auxiliary projections
- Axonomety
- Perspective

Types of problems

- Incidence and intersection problems, shadow constructions
- Metrical constructions
- Representation of spatial elements, polyhedrons, circle, sphere, cylinder and cone

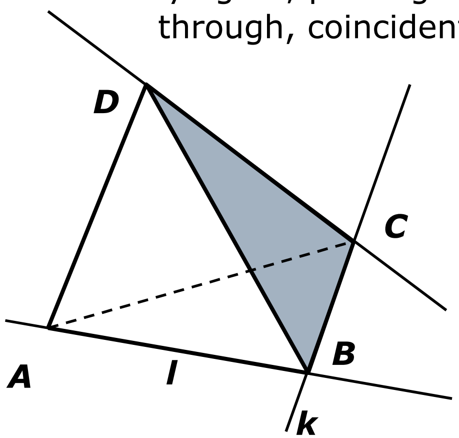
In Descriptive Geometry 1

We shall study

- representation of spatial elements and analyze their mutual positions
  - determine their angles and distances
- represent pyramids, prisms, regular polyhedrons,
  - construct the intersection of polyhedrons with line and plane, intersection of two polyhedrons
- construct shadows
  - cast shadow, self-shadow, projected shadow
- ***the principles of representation and solution of 3D geometrical problems in 2D***

Spatial elements, relations, notation

∥	parallel
∦	non parallel
⊥, ⊾	perpendicular
–o–	lying on, passing through, coincident



$\overline{AB}$	segment ***A,B***
|***AB***| = ***l***	line ***A,B***
B = ***k*** ∩ ***l***	intersection
α = [***BCD***]	plane ***B,C,D***
A –o̶– ***a***	not lying on

Relations

pair of points: determine a distance

point and line: lying on
- not lying on → plane, distance

pair of lines: coplanar
- intersecting → angles
- parallel → distance

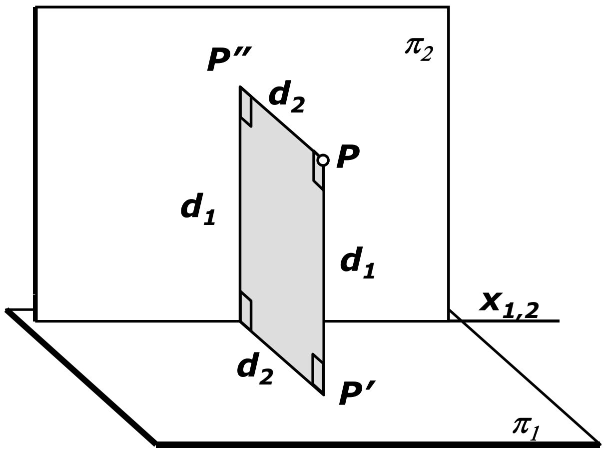
non coplanar
- skew → angle and distance

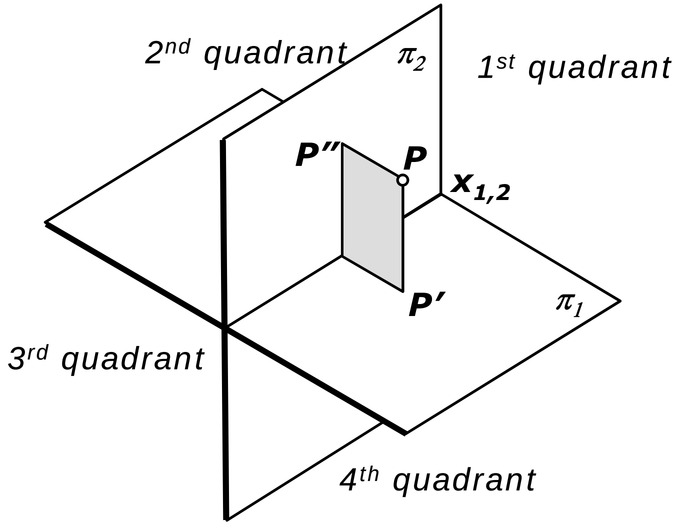
point and plane: lying on
- not lying on → distance

line and plane: parallel → distance
- intersecting → angle

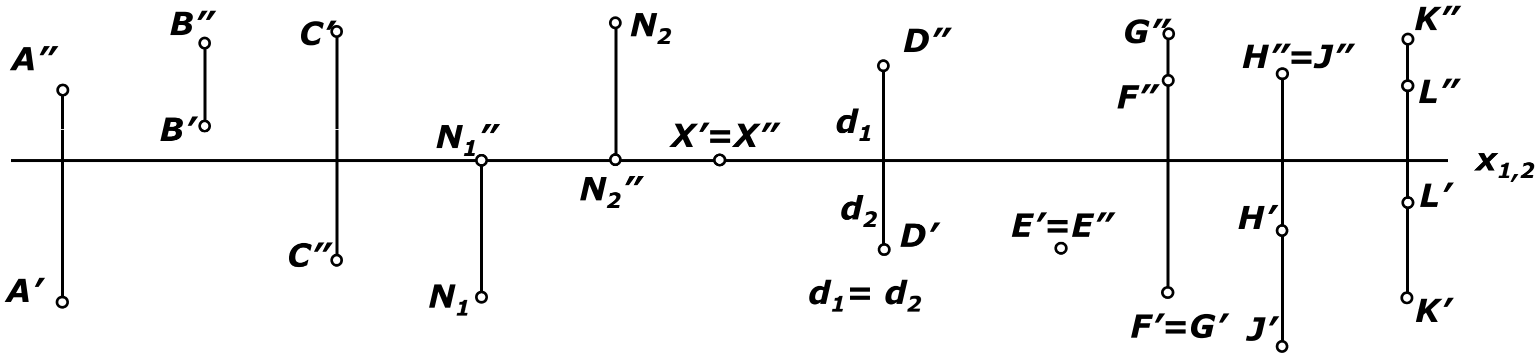
pair of planes: parallel → distance
- intersecting → angle

Representation of point



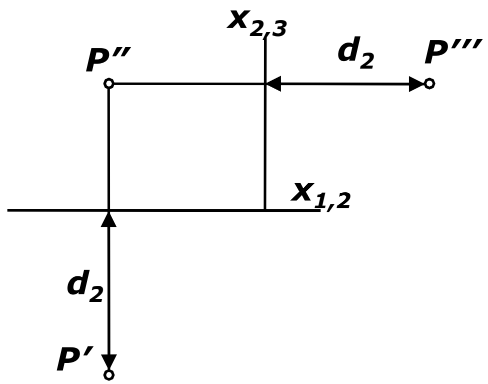


In which quadrant or image plane is the point located, why is it special?

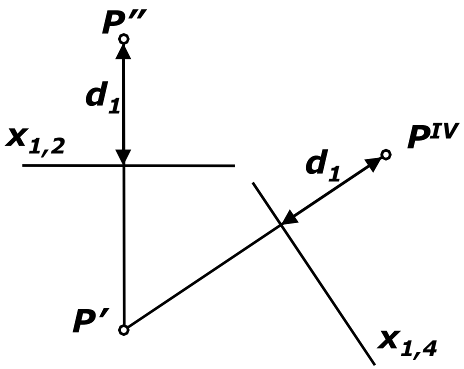


Auxiliary projections

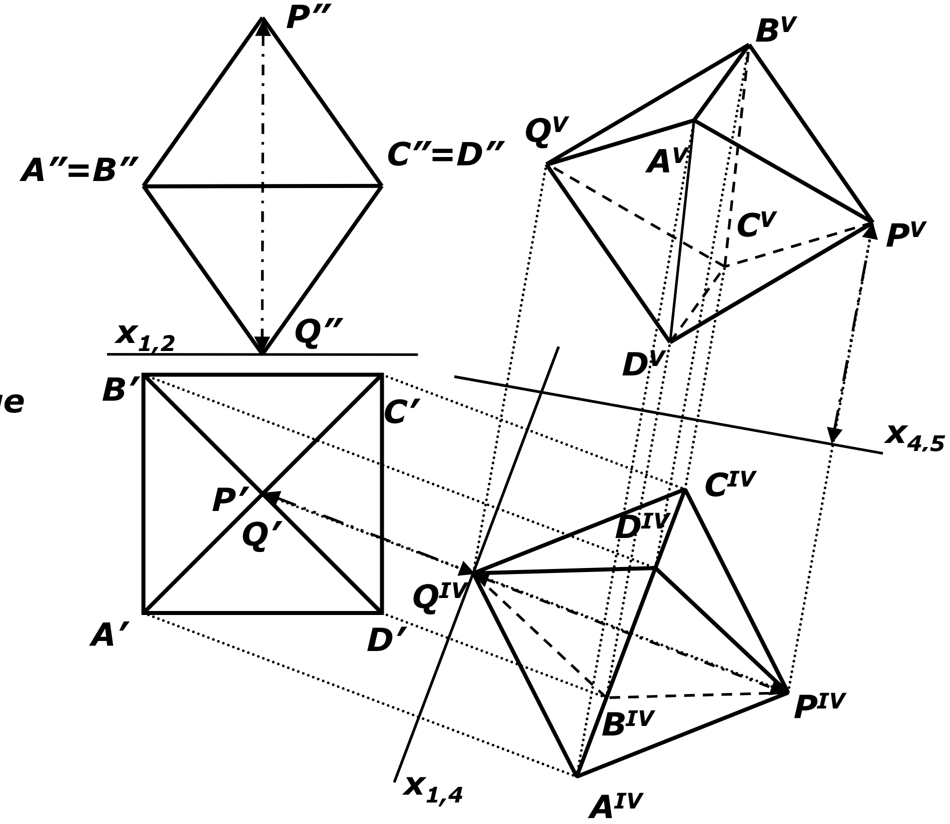
Side view, third image



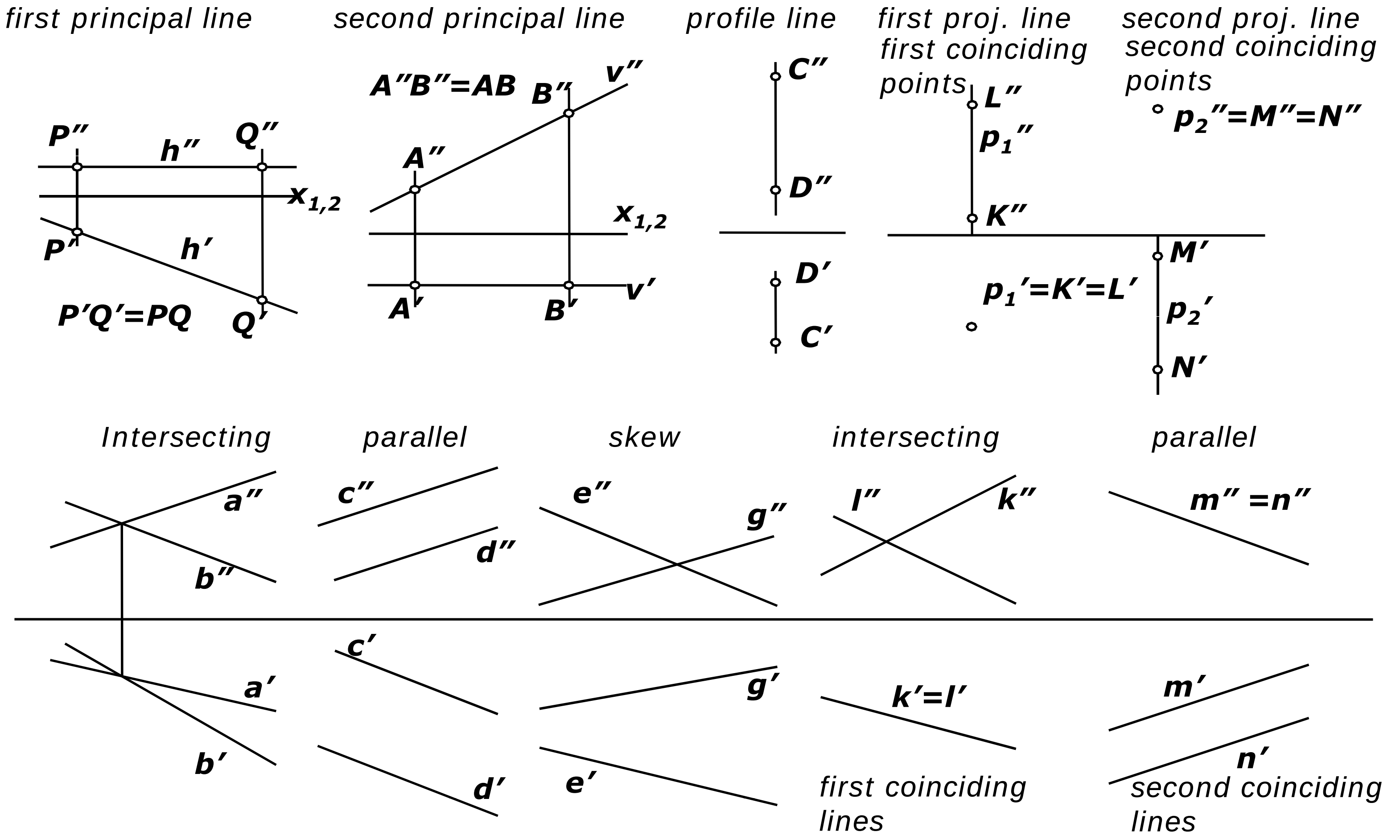
Fourth image, linked to the first image



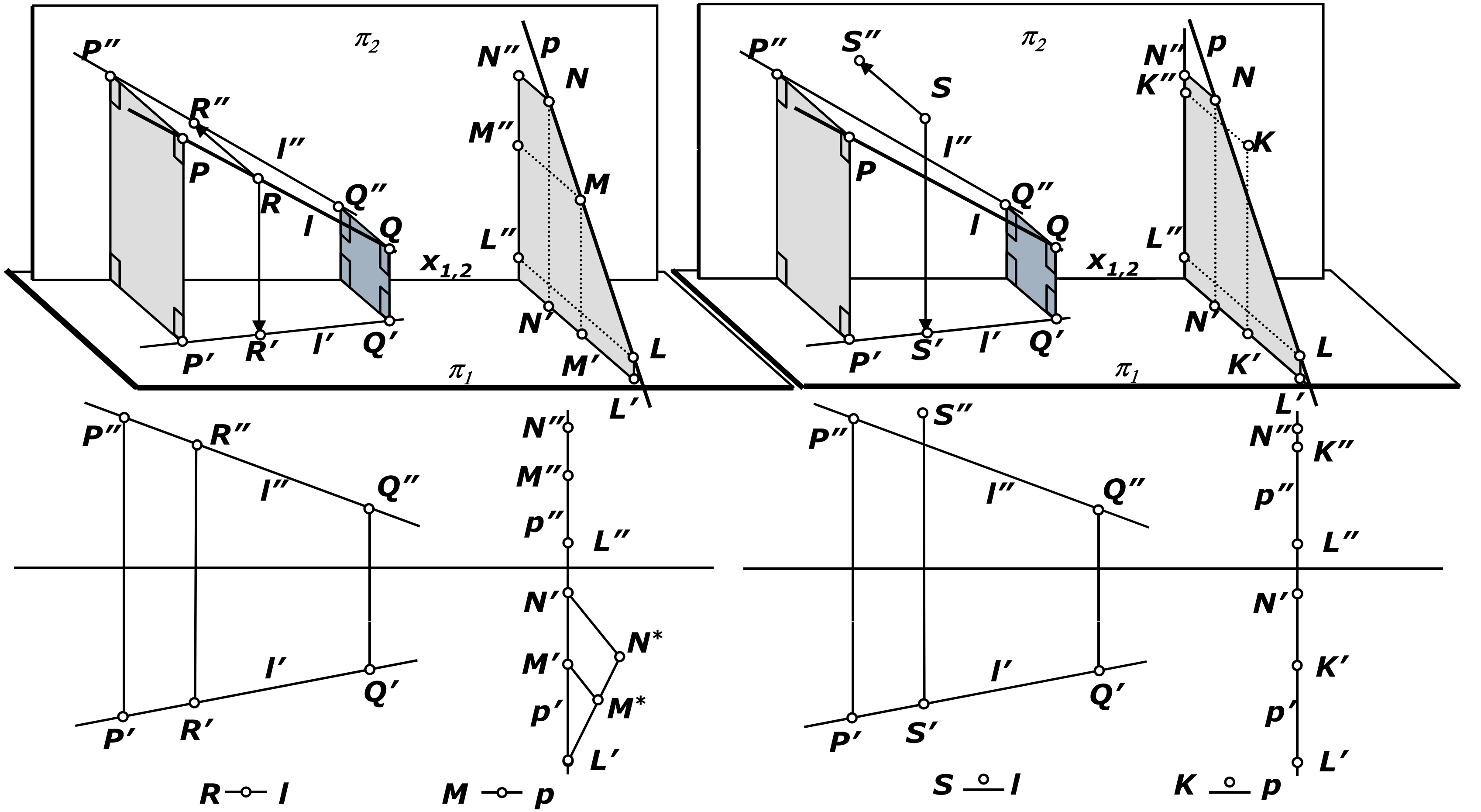
Chain of transformations



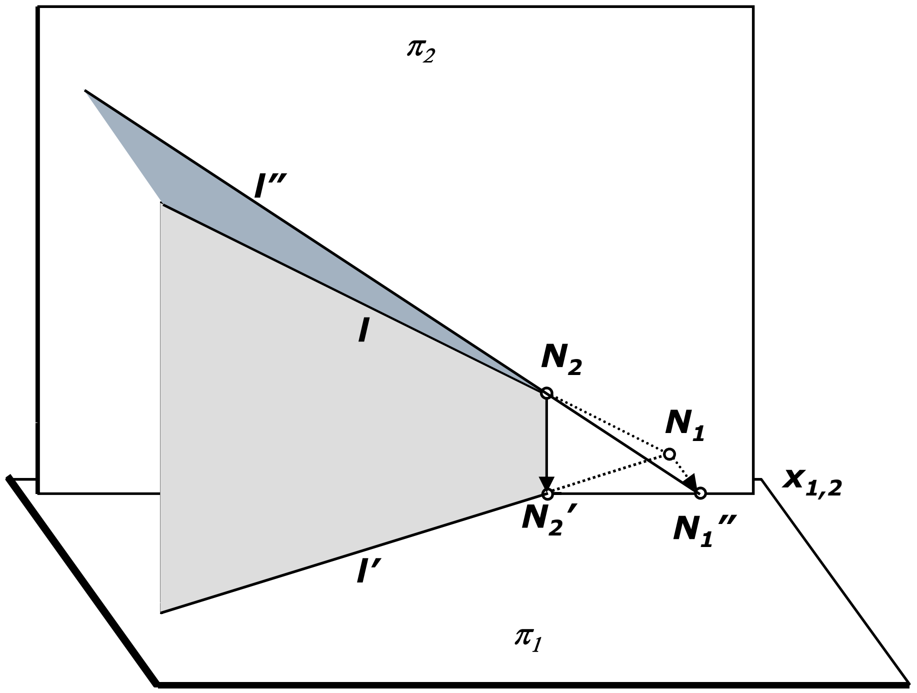
Representation of Straight Lines, Relative Positions

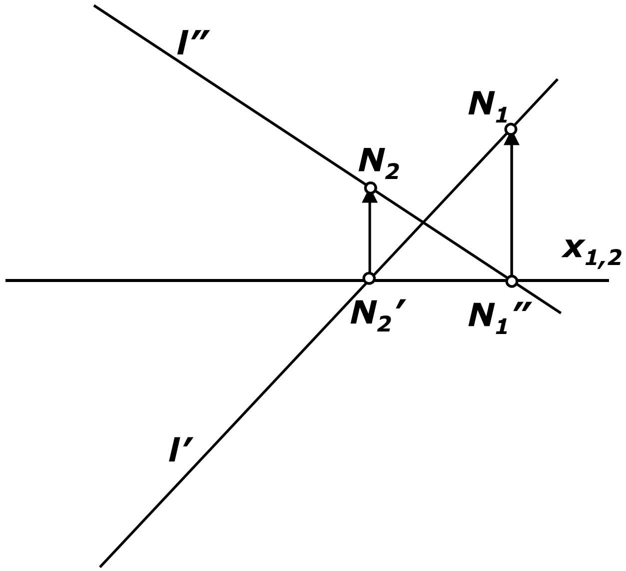


Point and Line



Tracing Points of a Line

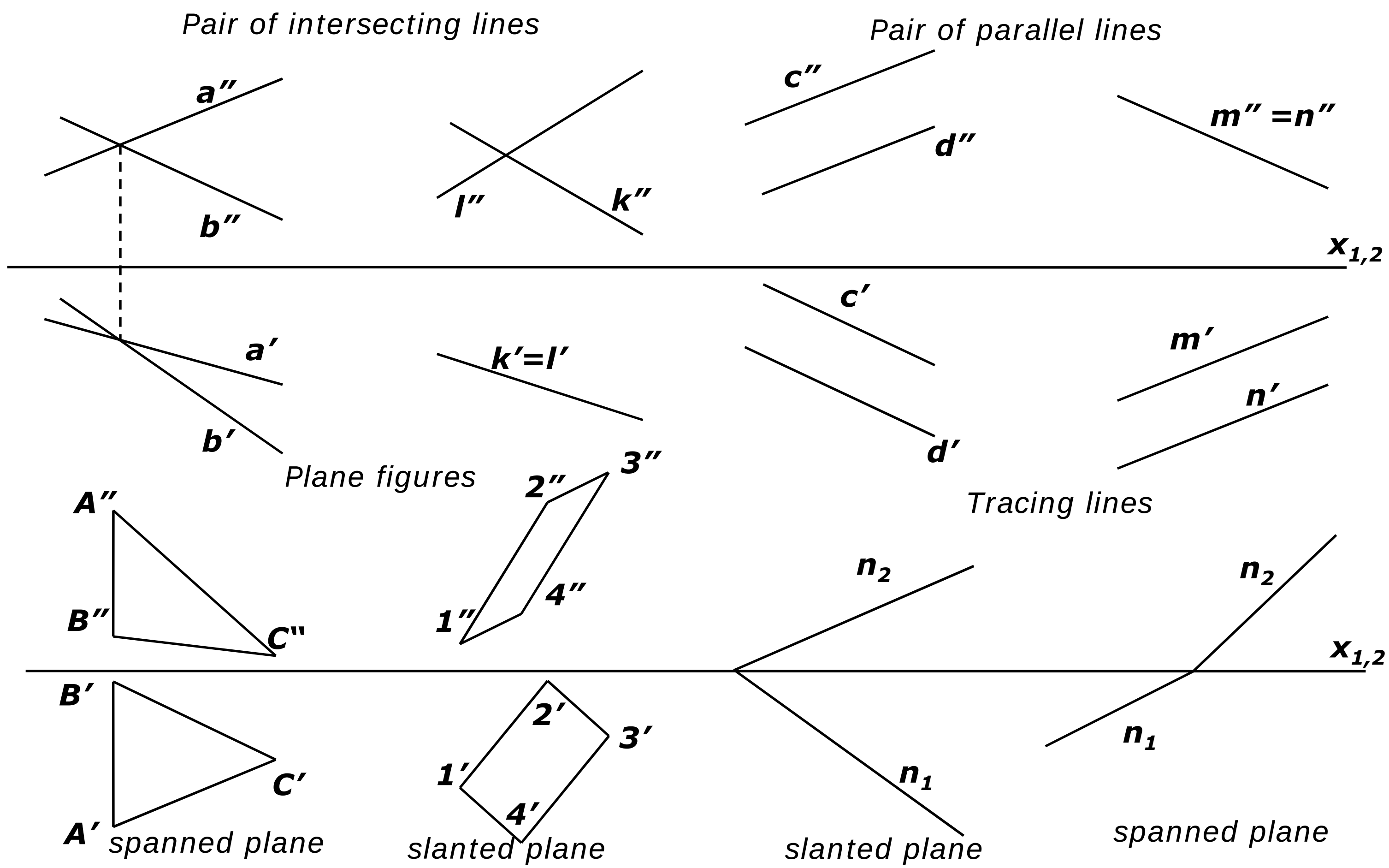




Problems:

1) *find the tracing points of principal /profile lines*
2) *determine lines by means of tracing points*

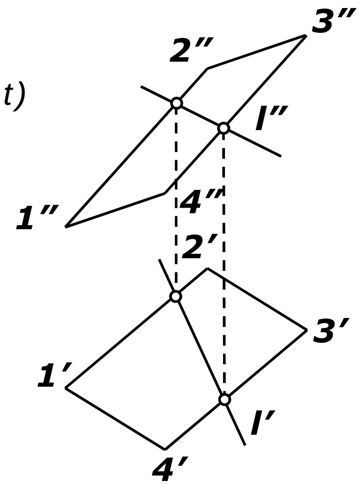
Representation of Plane



Line and Plane

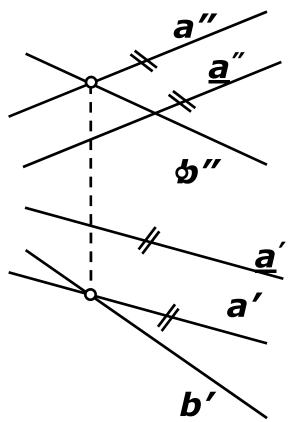
lying on (incident)

***l*–o–[*1234*]**



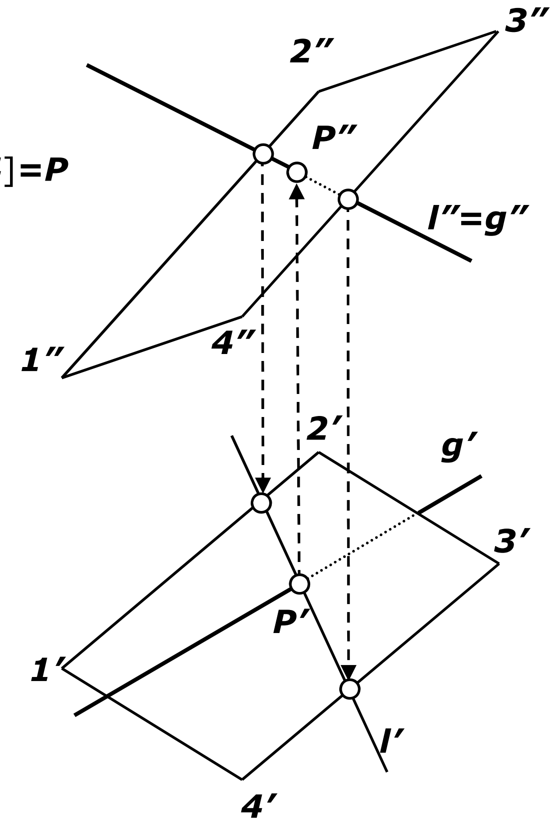
parallel

***a* ∥ [*ab*]**

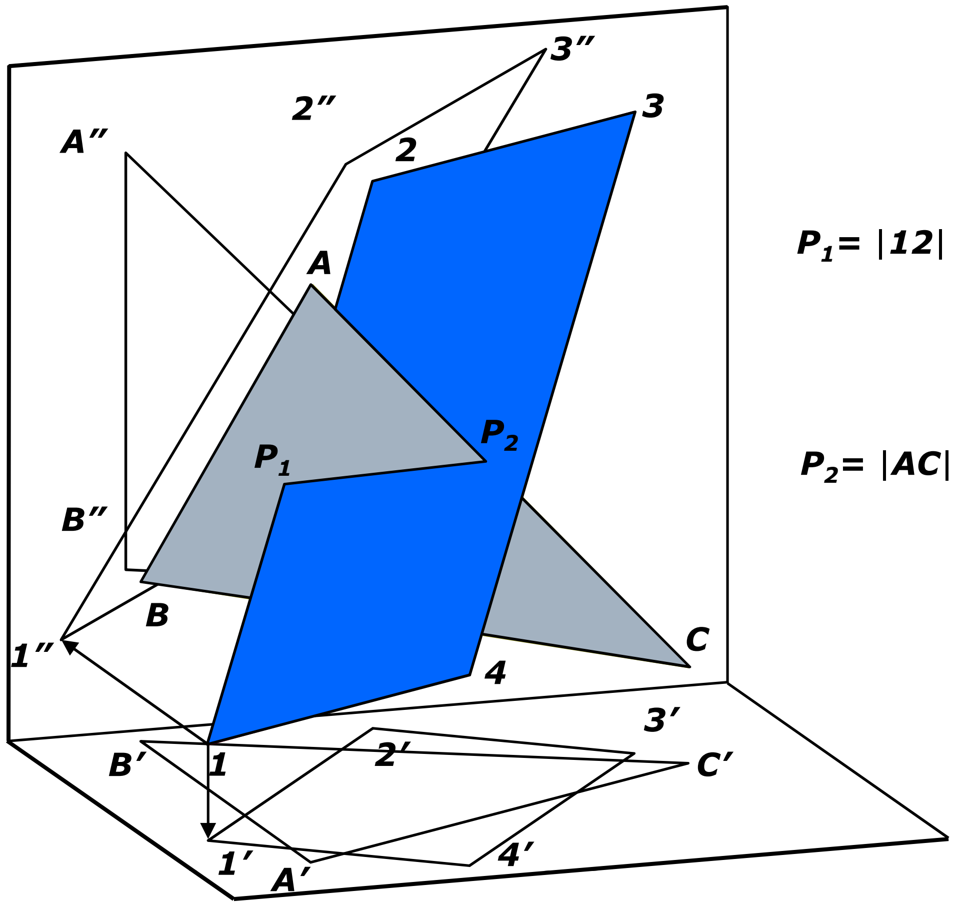


intersecting

g* ∩ [*1234*]=*P

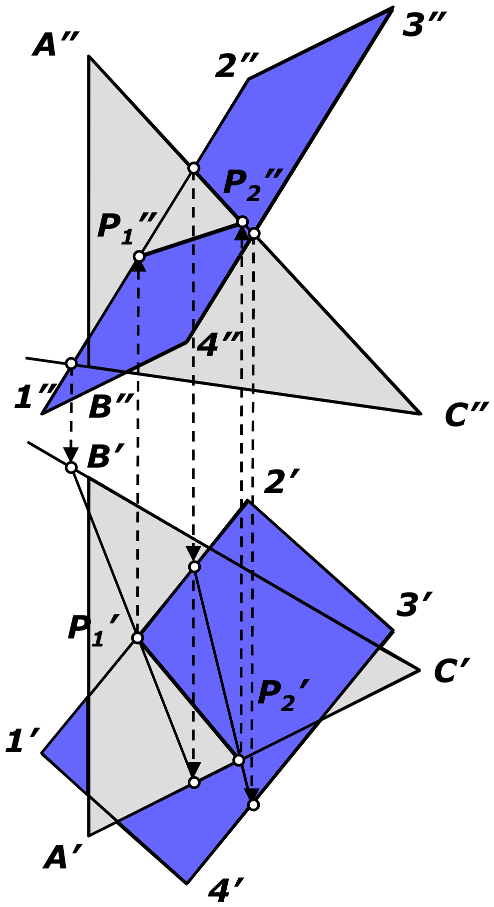


Intersection of Two Planes



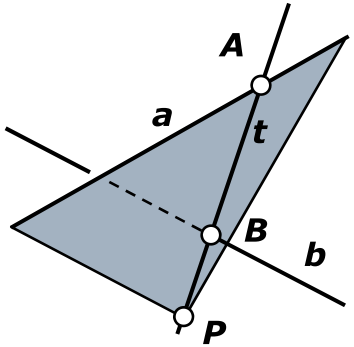
$P_1 = |12| \cap [ABC]$

$P_2 = |AC| \cap [1234]$



Transversal of a Pair Of Skew Lines Passing Through a Given Point

Sketch and algorithm



$B = [Pa] \cap b$

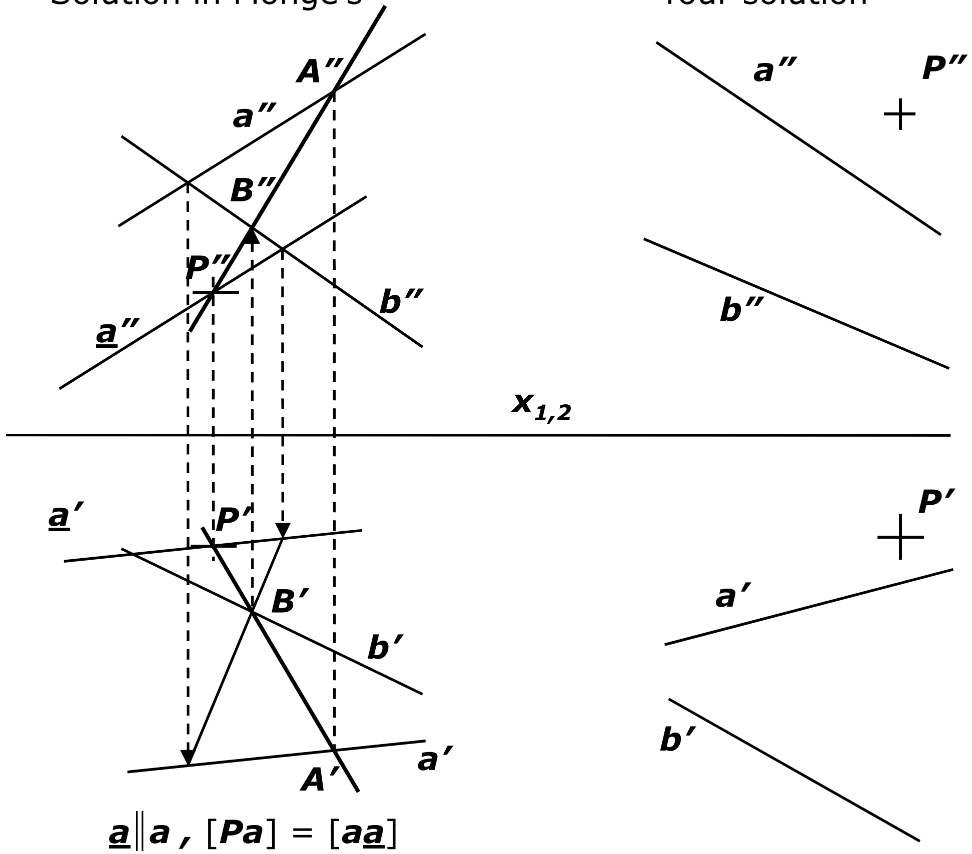
$t = |PB|$

or

$t = [Pa] \cap [Pb]$

Solution in Monge's

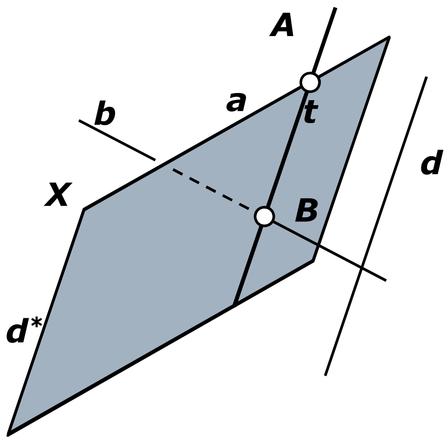
Your solution



$\underline{a} \parallel a$, $[Pa] = [a\underline{a}]$

Transversal of a Pair Of Skew Lines Parallel to a Given Direction

Sketch and algorithm



X −○− a

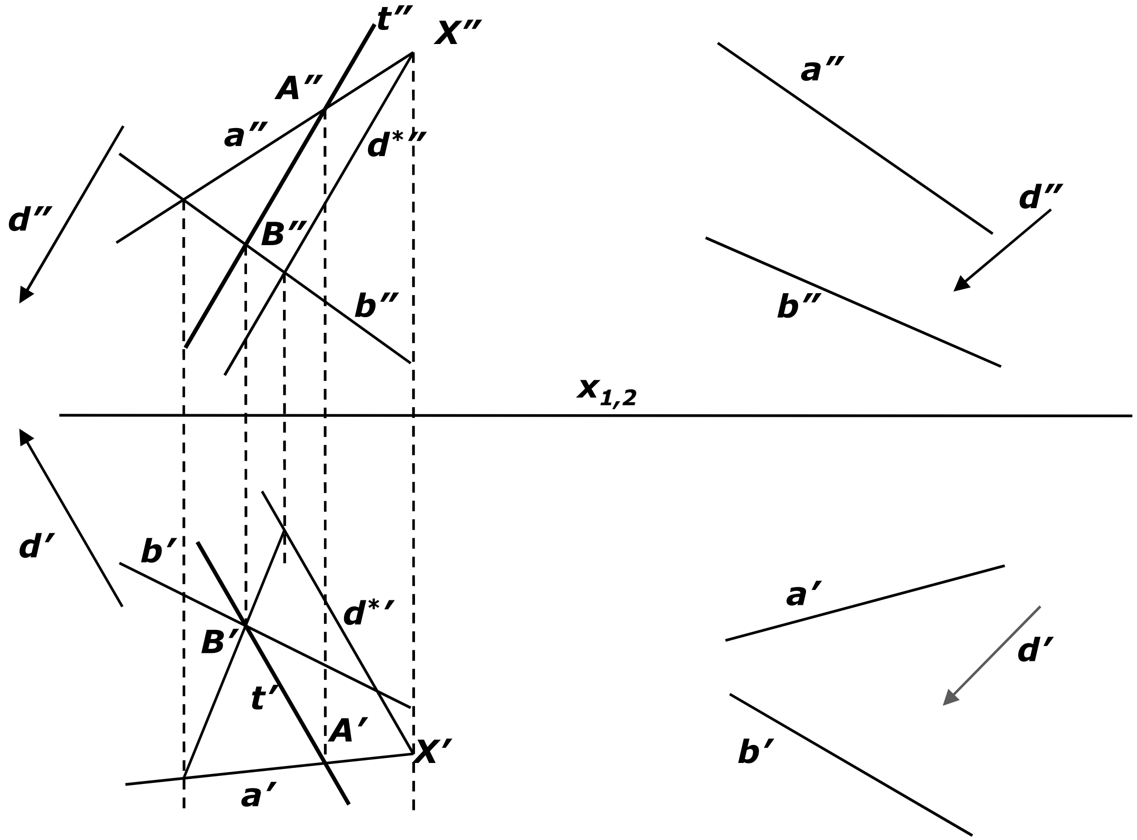
d*−○−X, d* ∥ d

B = b ∩ [ad*]

t−○−B, t ∥ d*

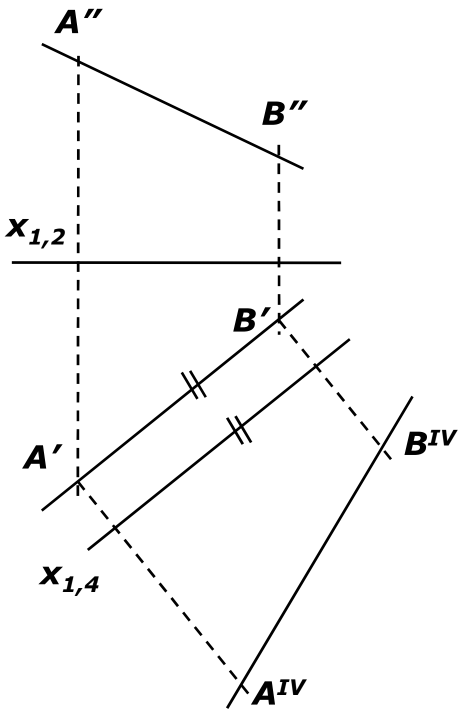
Solution in Monge's

Your solution

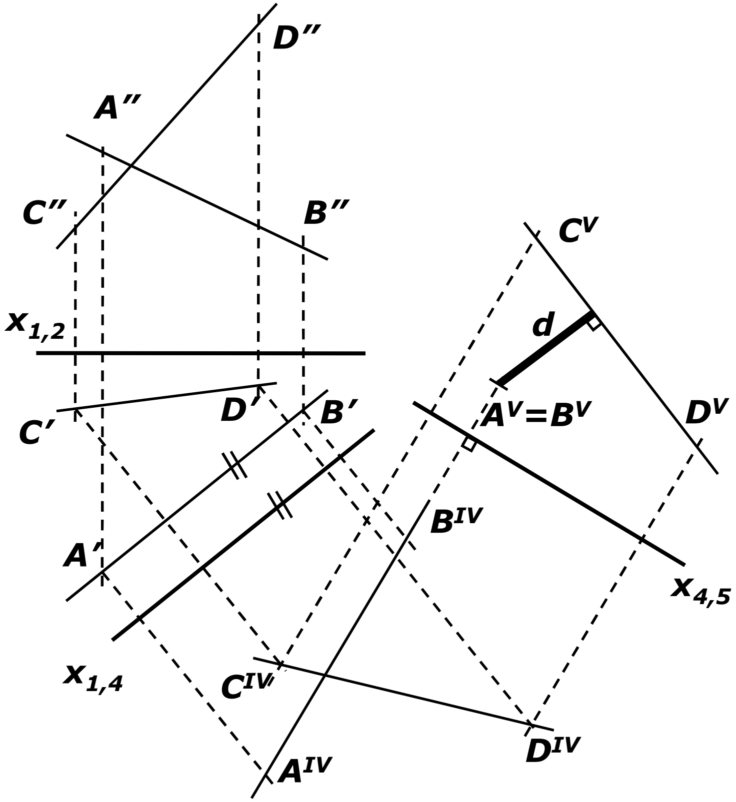


Auxiliary Projections on Special Purposes 1

True length of a segment



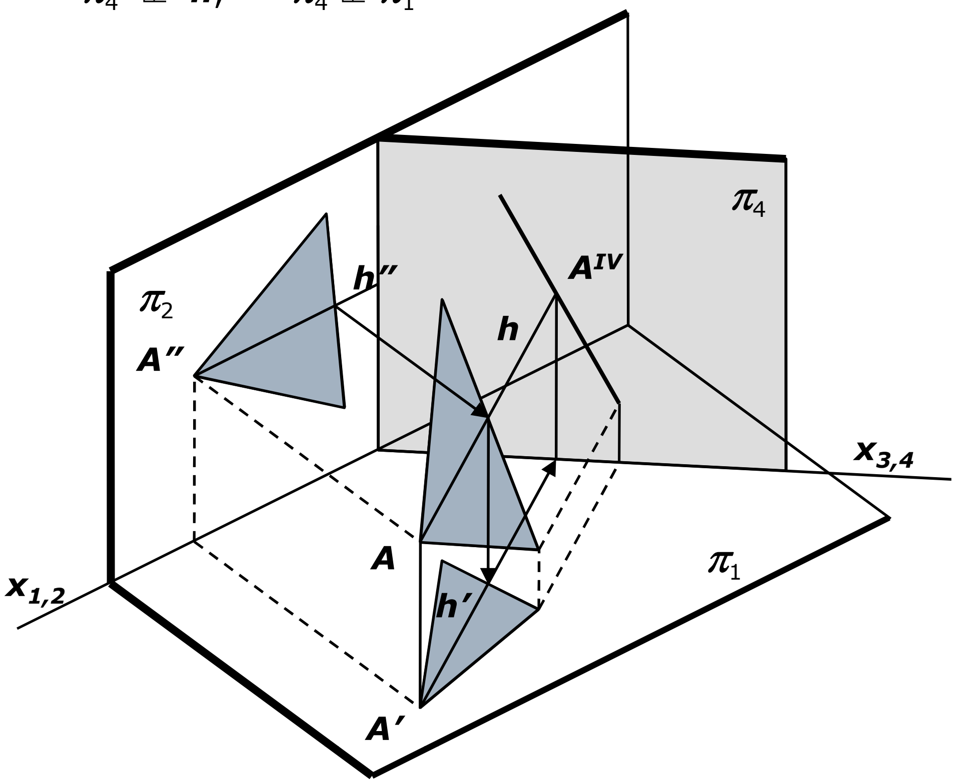
Distance of a pair of skew lines



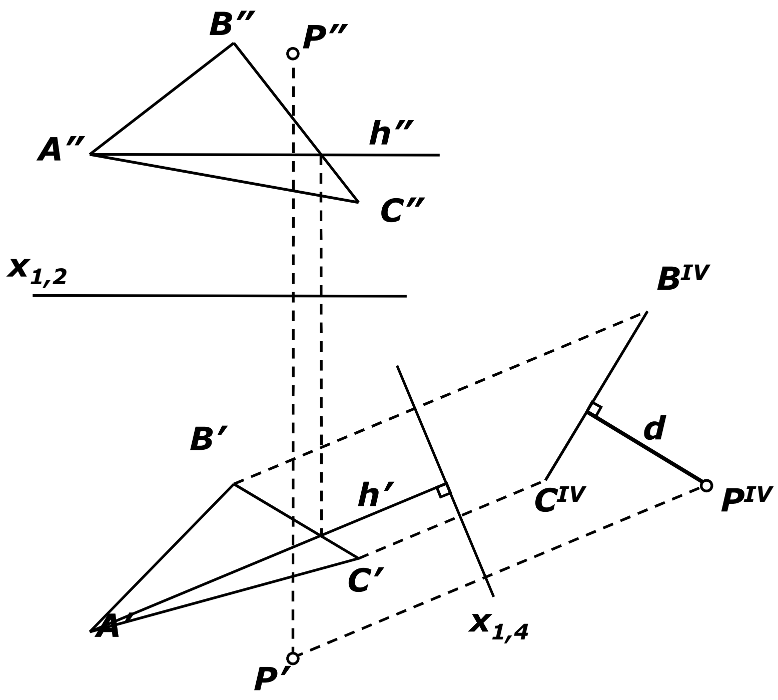
Auxiliary Projections on Special Purposes 2

Edge view of a plane: transformation of a plane in projecting plane

$\pi_4 \perp h, \quad \pi_4 \perp \pi_1$



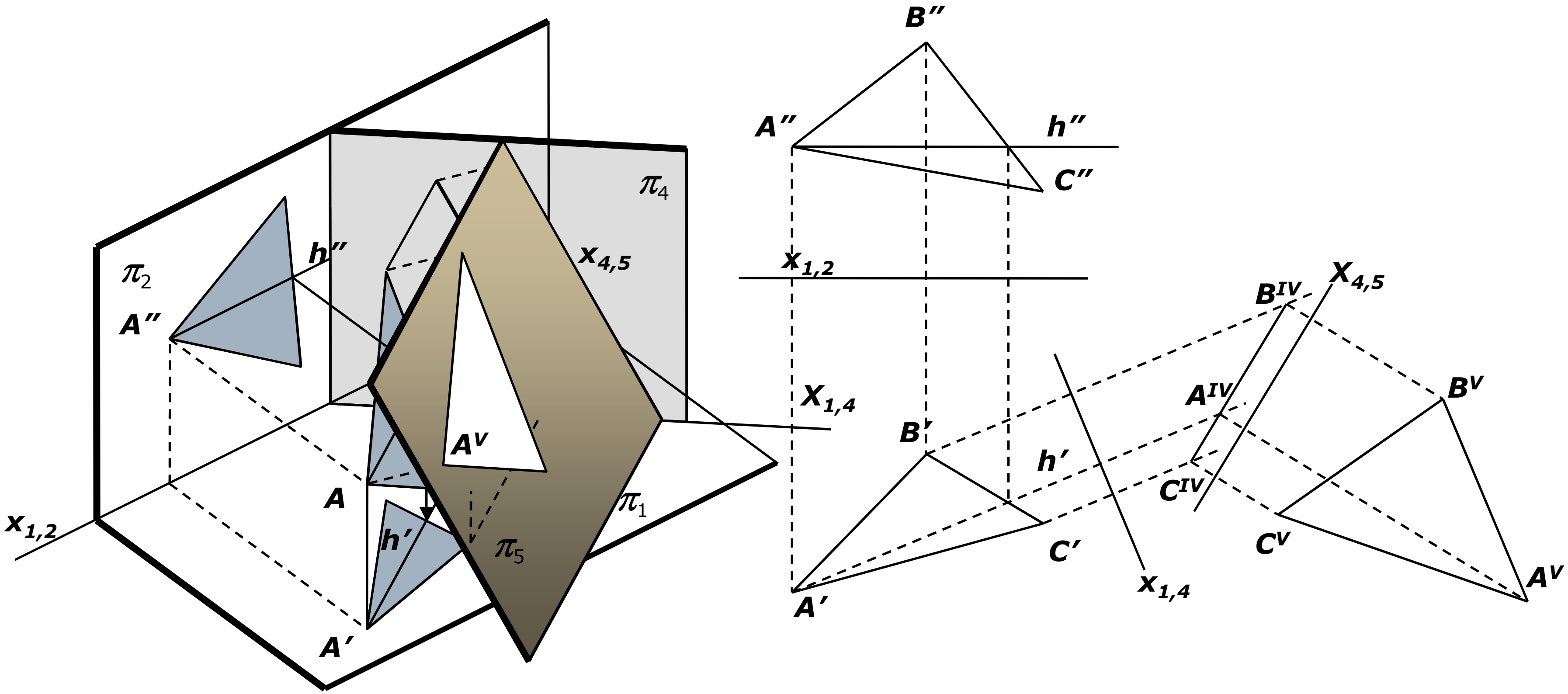
Application: find the distance ***d*** of the point ***P*** and the plane [***ABC***].



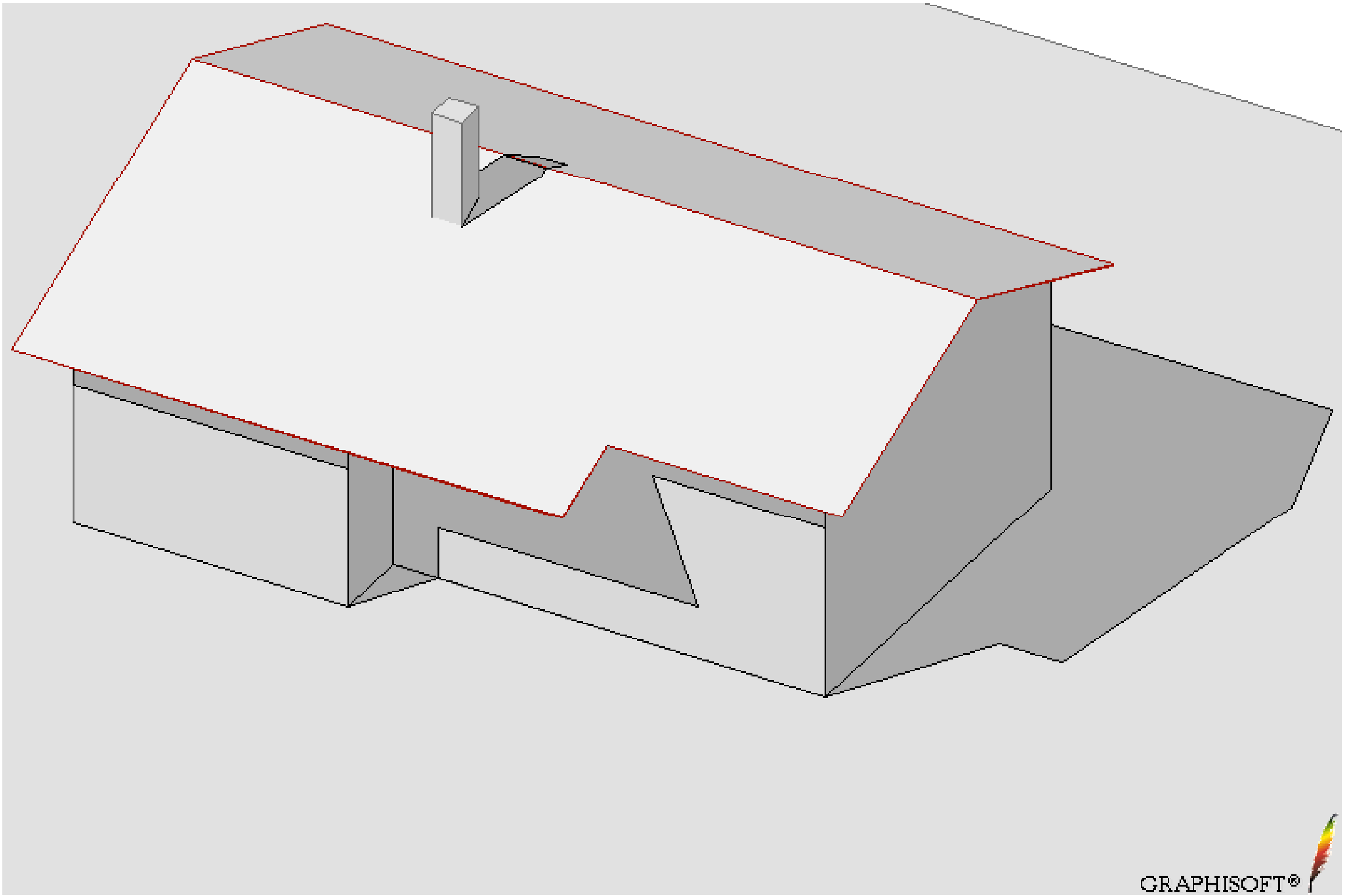
Auxiliary Projections on Special Purposes 3

Construction of the true shape of a figure lying in a general plane

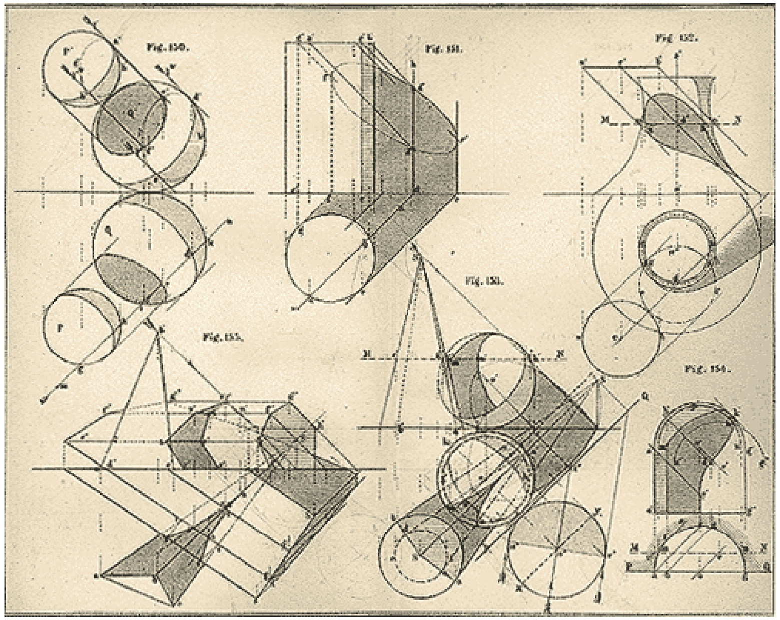
General plane → fourth projecting plane → fifth principal plane



Cast Shadow, Self-shadow, Projected Shadow



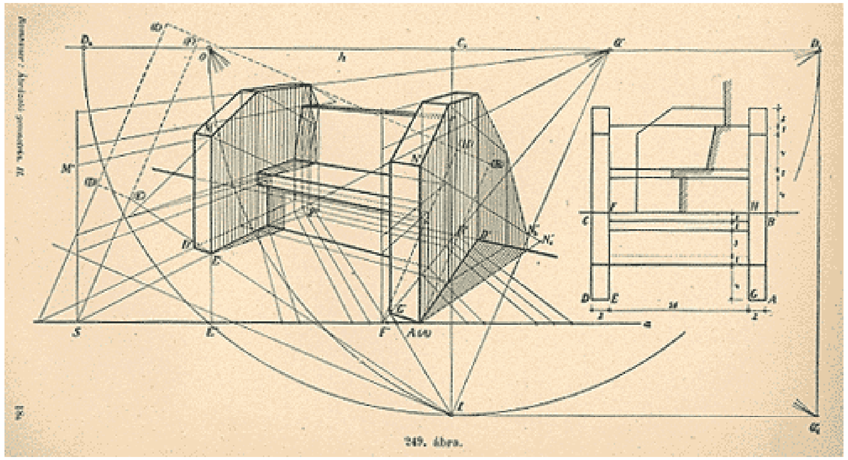
Shadow in Traditional Descriptive Geometry



Riess, C.: Grundzüge der darstellenden Geometrie

(Stuttgart : Verl. J. B. Metzleráschen Buchhandlung, 1871)

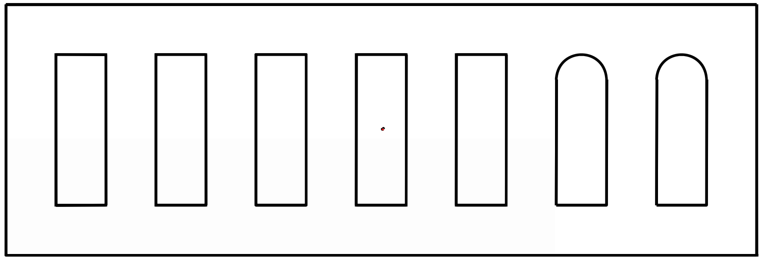
Application of Descriptive Geometry for Construction of Projected Shadow (plate X.)

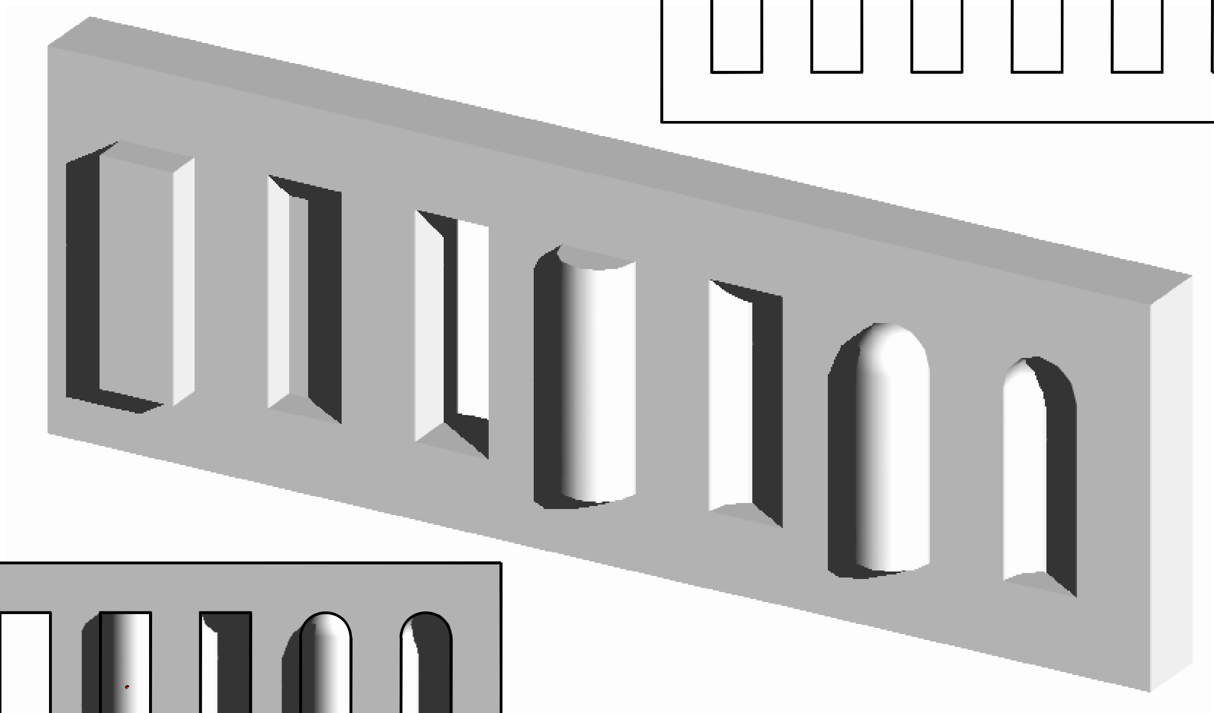


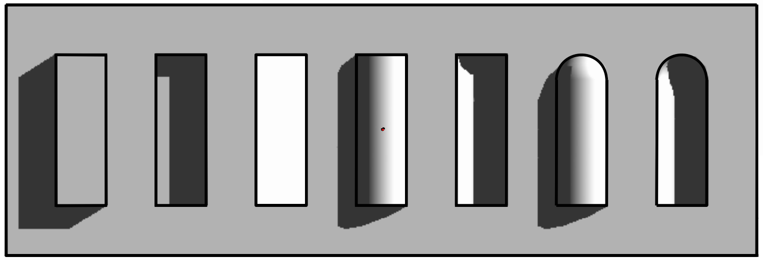
Romsauer Lajos: Ábrázoló geometria
(Budapest : Franklin-Társulat, 1929)

http://www.c3.hu/perspektiva/adatbazis/

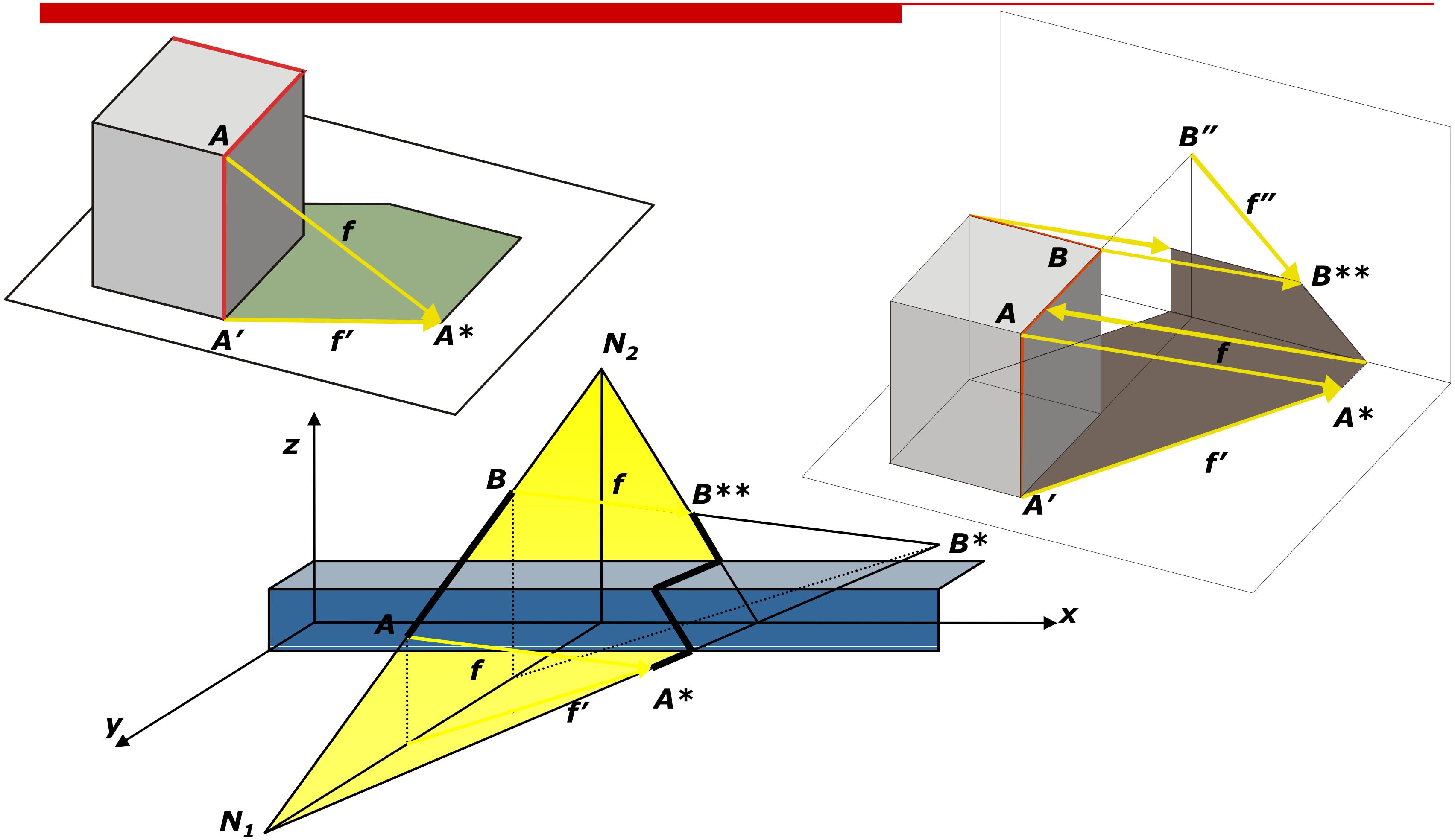
Shadow in Visualisation







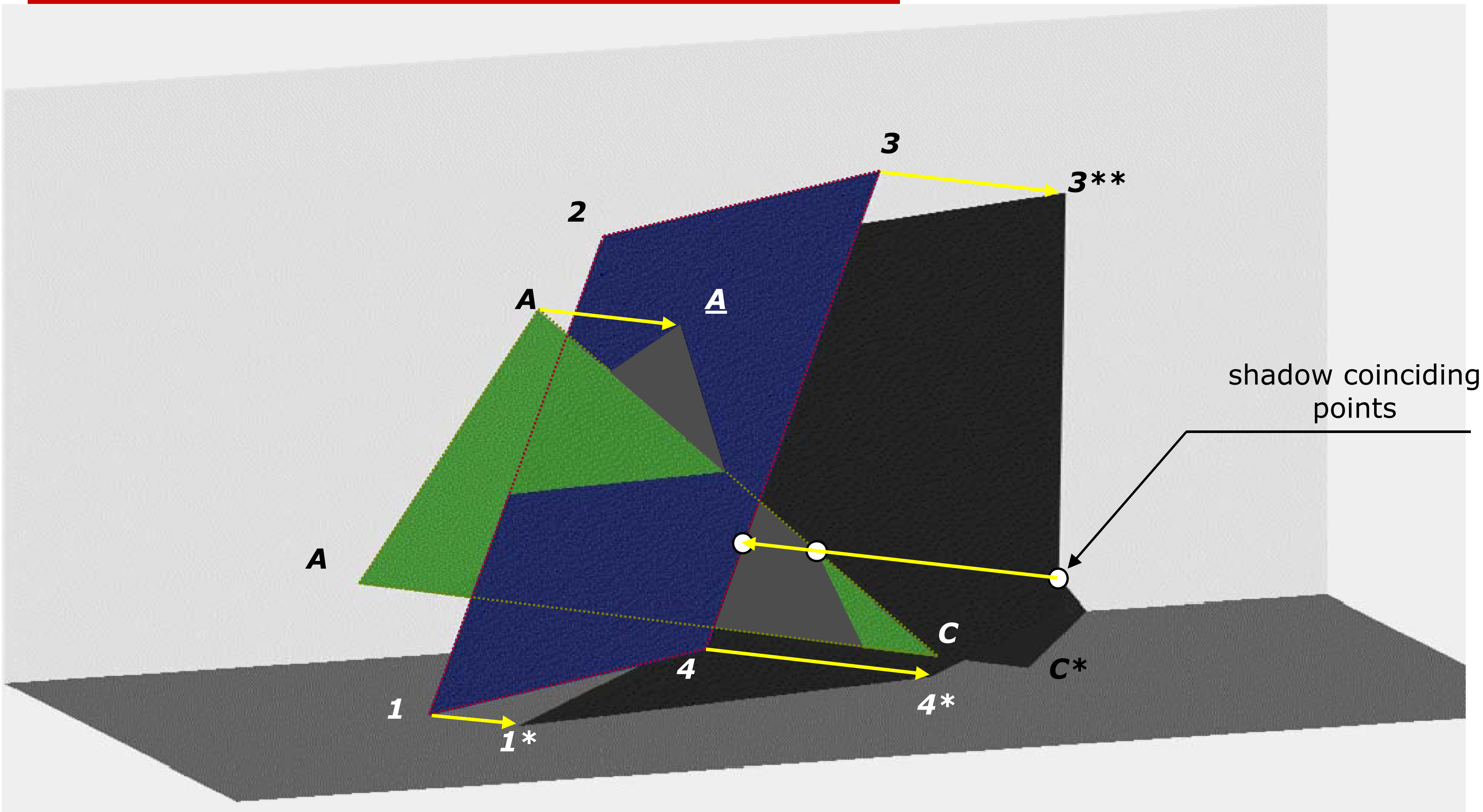
Shadows - Basics



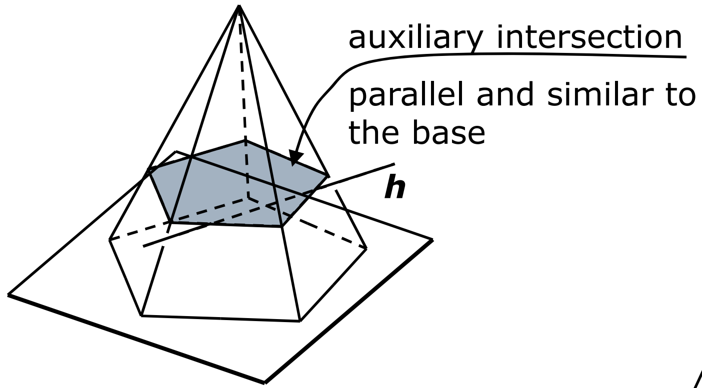
Shadow Properties

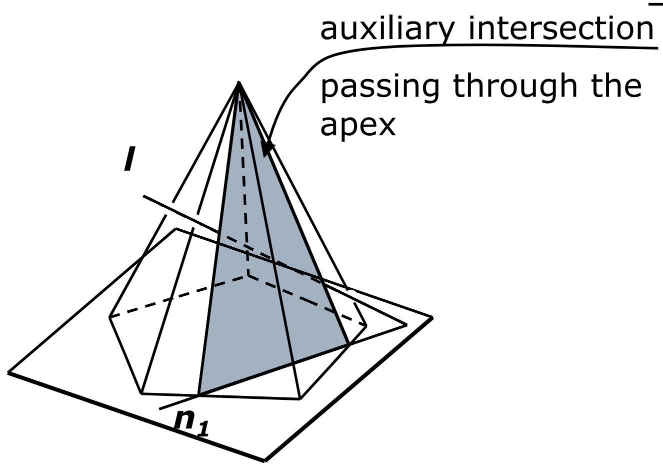
1) Our constructions are restricted to parallel lighting.
2) We do not represent transition between dark and light shade.
3) We usually construct three types of shadow: cast shadow on the ground or on the image planes, self-shadow (shade) and projected shadow.
4) Shadow of a point: piercing point of the ray of light passing through the point, in the surface (on ground plane, picture plane etc.)
5) Shadow of a straight line: intersection of the plane passing through the line, parallel to the direction of lighting and the surface (screen).
6) Shadow of a curve: the intersection of cylinder (whose generatrix is the curve, the generators are rays of light) with the surface (screen).
7) Shadow-coinciding points: pair of distinct points, whose shadows coincide.
8) Alongside cast shadow the surface is in self-shadow.
9) In case of equal orientation of a triangle and its shadow, the face of triangle is illuminated.
10) The cast shadow outline is the shadow of the self-shadow outline.

Cast Shadow, Projected Shadow

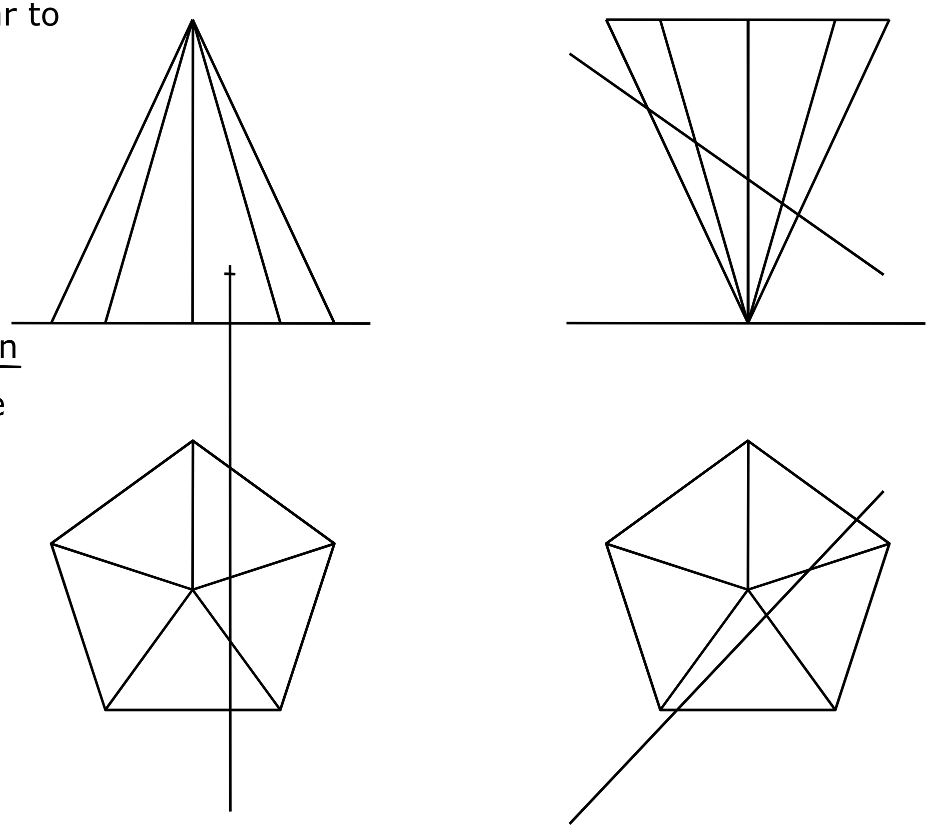


Intersection of Pyramid and Line

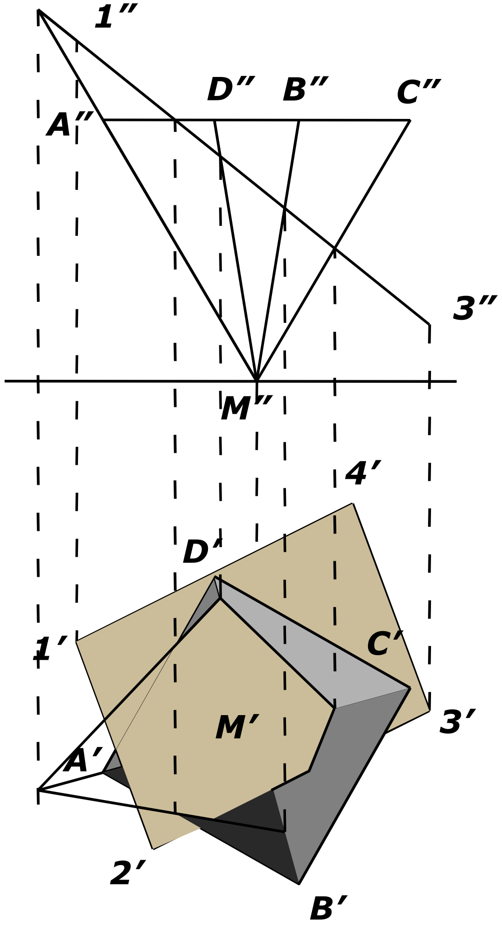




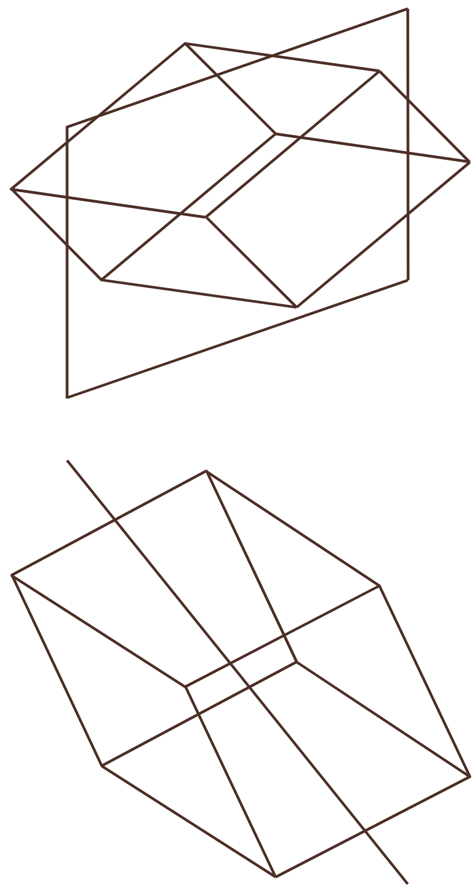
Find the intersection of line and pyramid



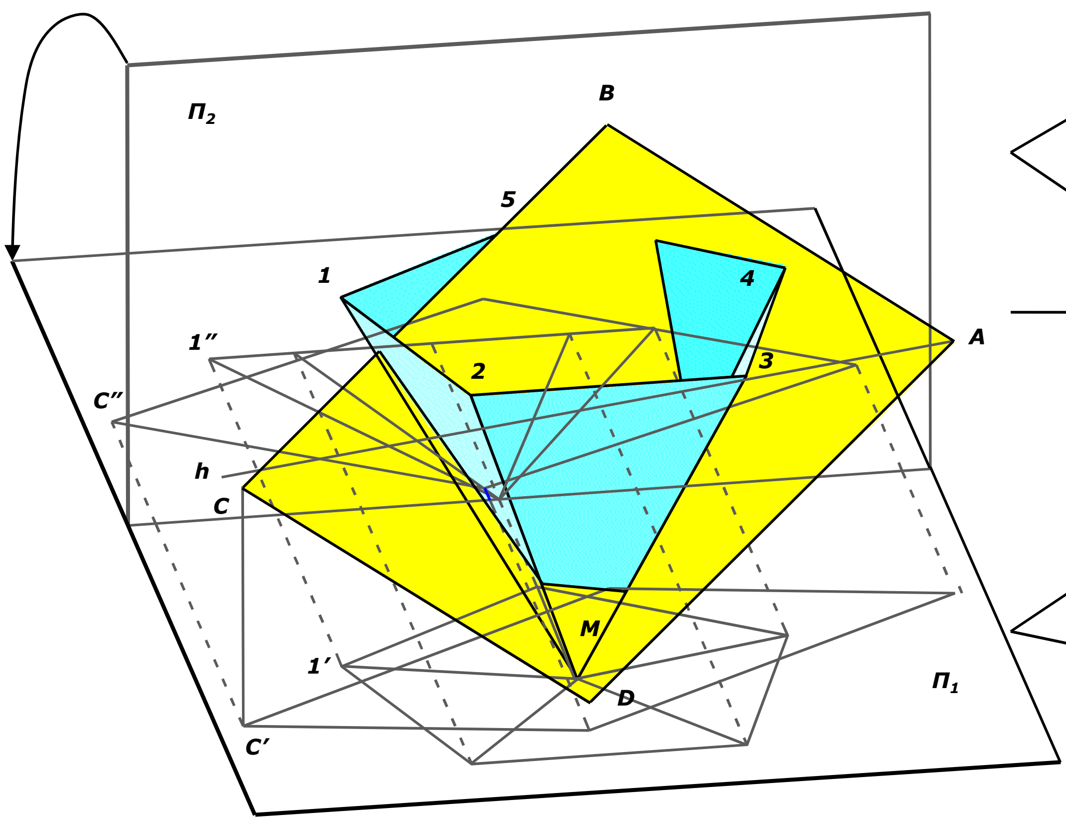
Intersection of Polyhedron and Projecting Plane

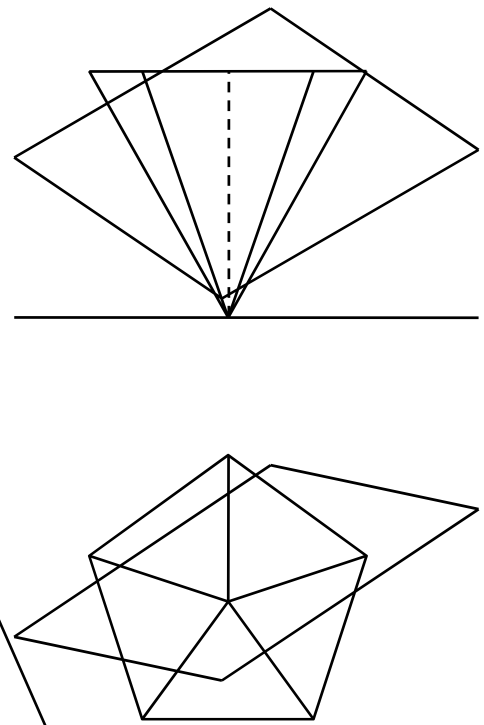


Find the intersection of plane and polyhedron



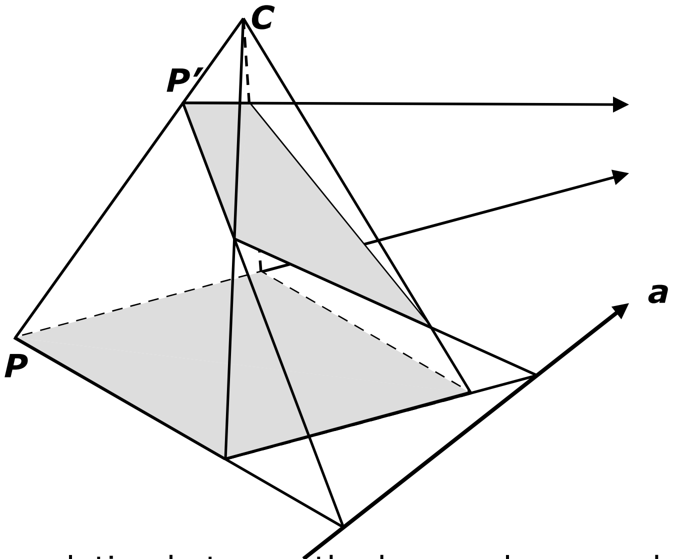
Intersection of Polyhedron and Plane (auxiliary projection)

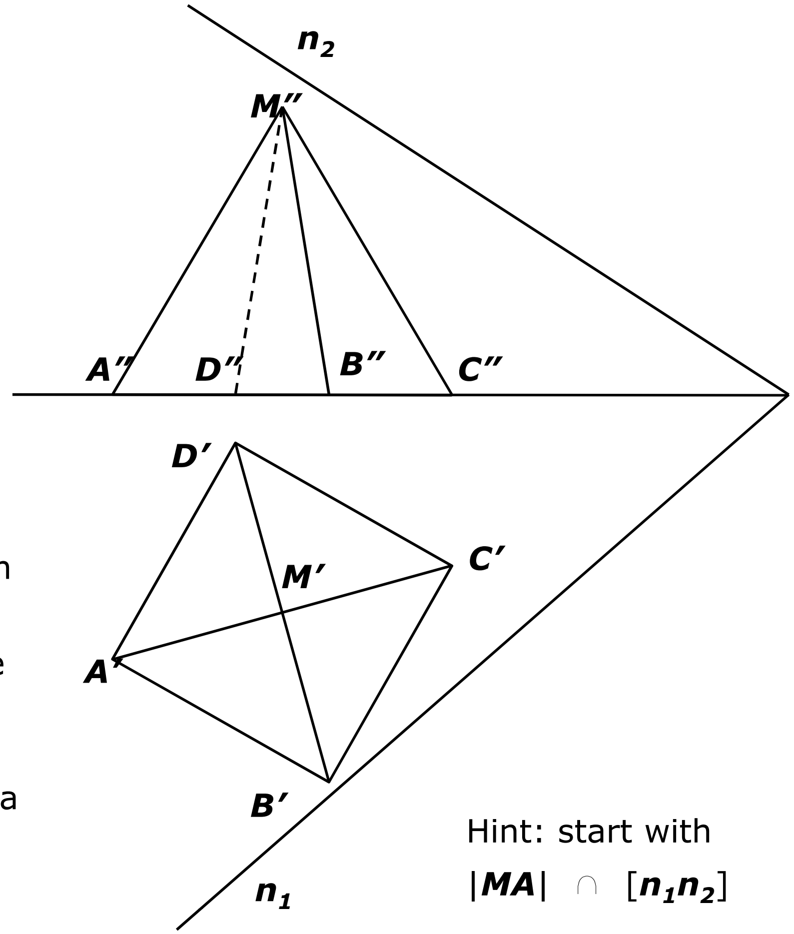




Hint: introduce $\boldsymbol{\Pi}_4$ image plane perpendicular to $\boldsymbol{\Pi}_1$ and the plane of parallelogram ($\boldsymbol{\Pi}_4 \perp \boldsymbol{h} \rightarrow \boldsymbol{x_{1,4}} \perp \boldsymbol{h'}$).

Intersection of Pyramid and Plane (Collineation)





The relation between the base polygon and the polygon of intersection is ***central-axial collineation***.

The **axis** of collineation is the line of intersection of the base plane and the plane of intersection, the center is the apex of the pyramid.

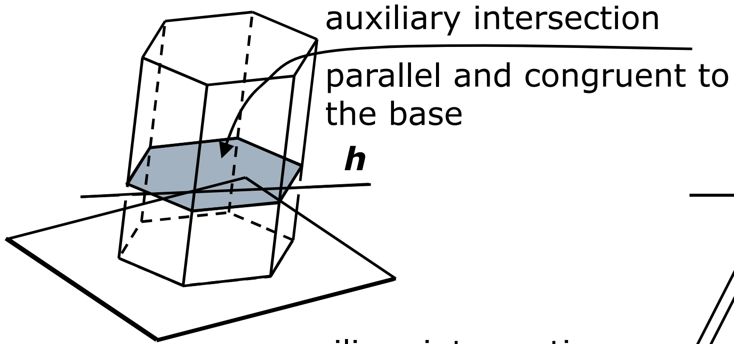
A **pair of corresponding points** is the pedal point of a lateral edge and the piercing point of the edge in the plane of intersection.

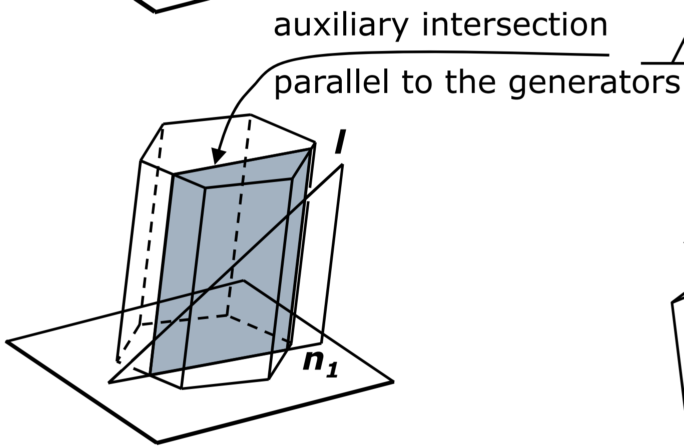
The **center** is the vertex of the pyramid.

Hint: start with

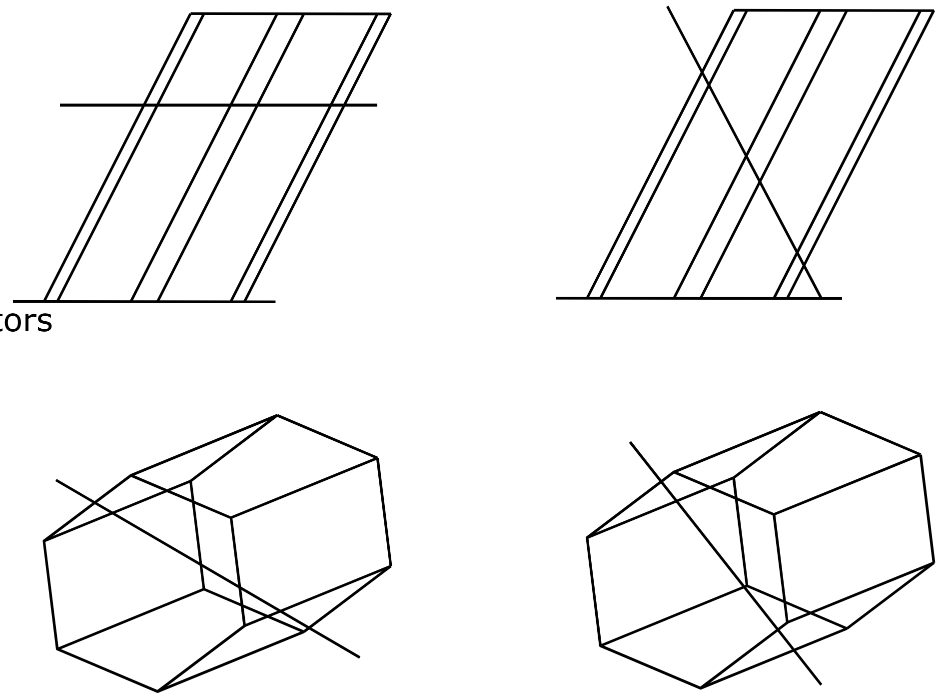
$|MA| \cap [n_1 n_2]$

Intersection of Prism and Line

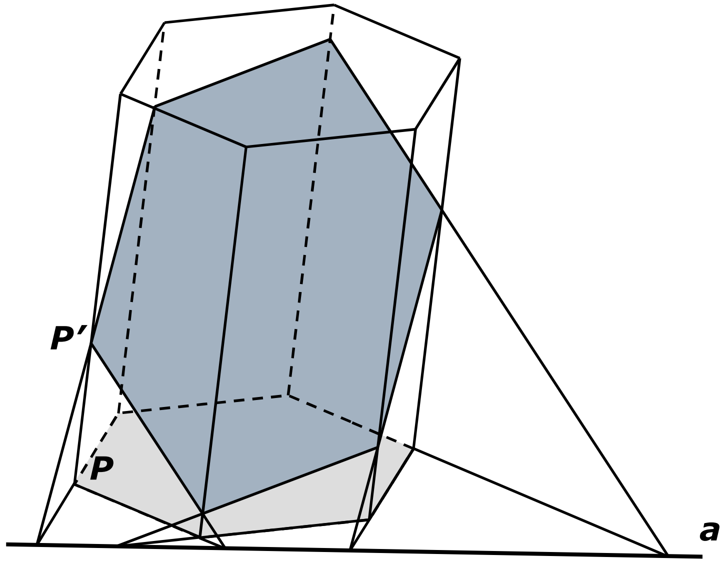


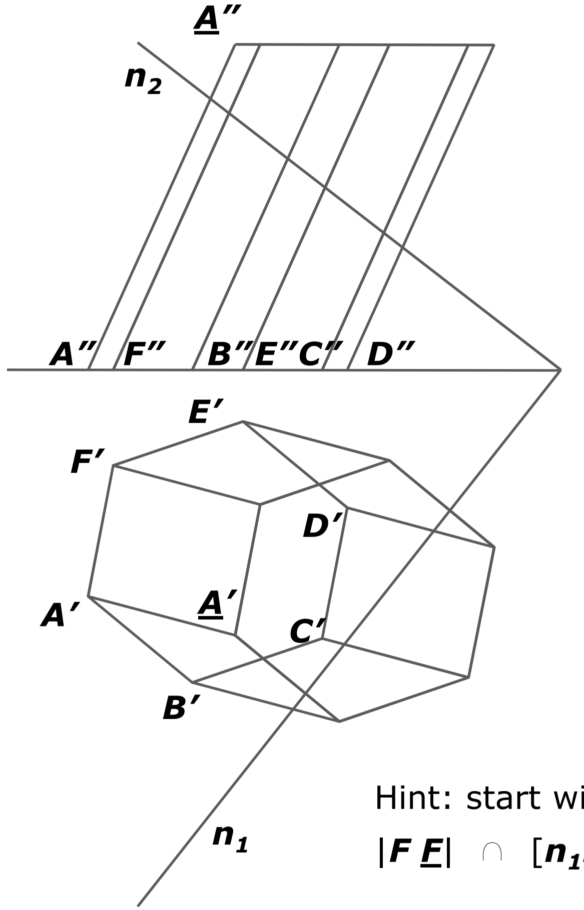


Find the intersection of line and prism



Intersection of Prism and Plane (affinity)





The relation between the base polygon and the polygon of intersection is ***axial affinity***.

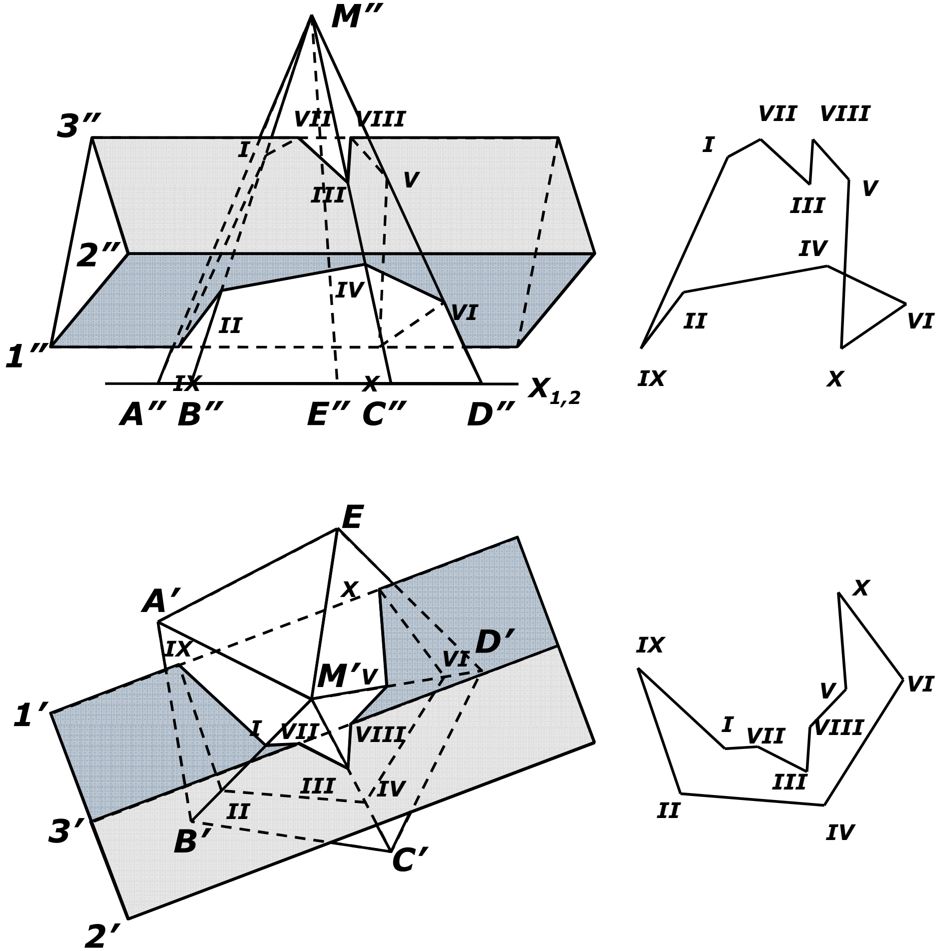
The **axis** of affinity is the line of intersection of the base plane and the plane of intersection.

A **pair of corresponding** points is the pedal point of a lateral edge and the piercing point of the edge in the plane of intersection.

Hint: start with

$|F\ \underline{F}| \cap [n_1 n_2]$

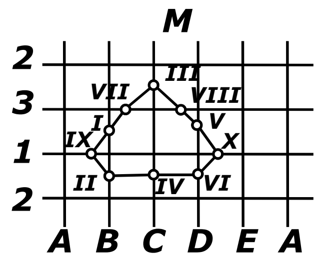
Intersection of a Pair of Solids



The intersection of two polyhedrons is a polygon (usually 3D polygon).

The vertices of the polygon of intersection are the piercing points of the edges of a polyhedron in the faces of the other polyhedron.

The edges of the polygon of intersection are segments of intersection of pairs of faces.



Sequence: ***I-VII-III-VIII-V-X-VI-IV-II-IX-I***

At the visibility, one can think of solids or surfaces.

The visibility depends on, what we want to represent as a result of set operation: union, intersection or a kind of difference.

Intersection of a Pair of Solids (your solution)

Algorithm:

Introduce auxiliary image plane perpendicular to the horizontal edges of the prism

Construct the fourth image

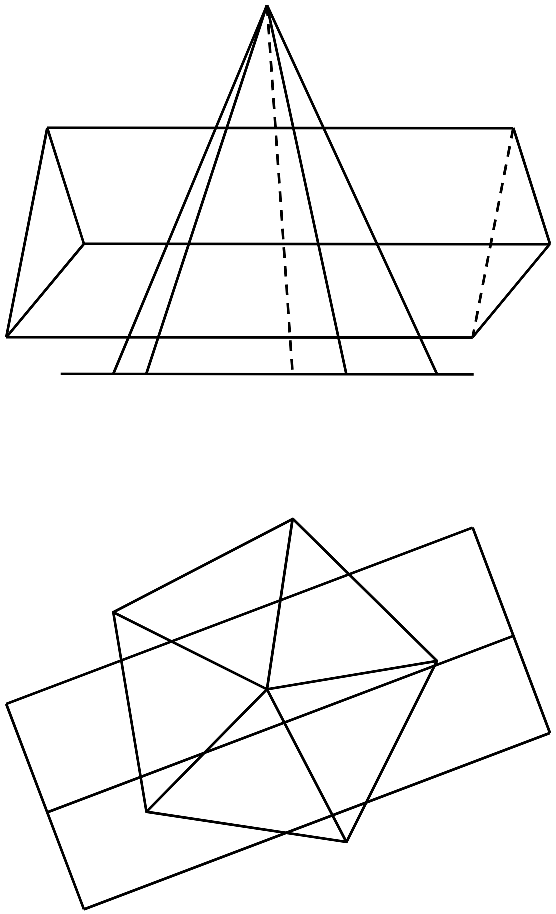
Find the piercing points of the edges of pyramid in the faces of the prism

Find the piercing points of the edges of prism in the faces of the pyramid

Find the right sequence of the vertices of polygon of intersection

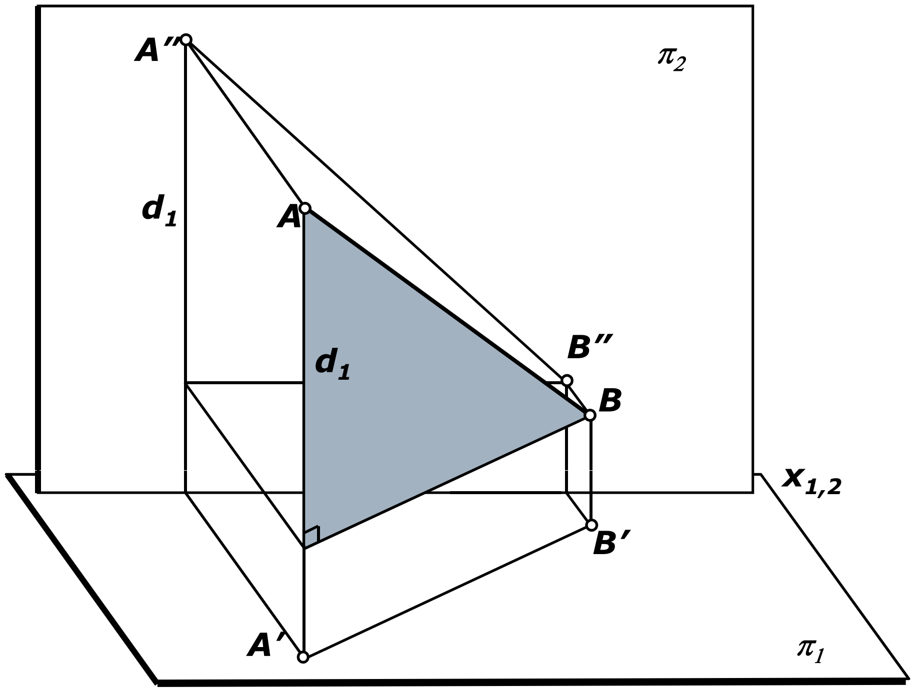
Draw the polygon of intersection in both images

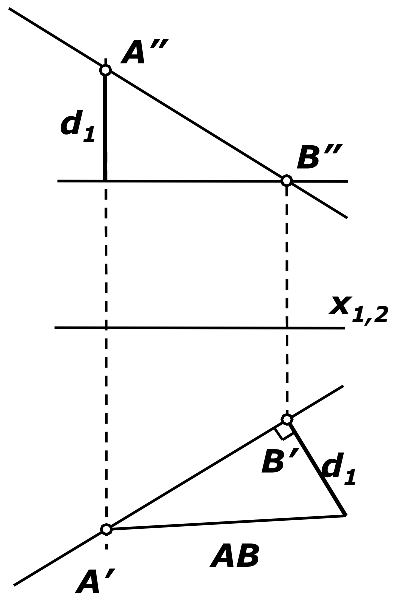
Show the visibility



Basic Metrical Constructions 1

The true length of a segment is the hypotenuse of right triangle. One of the legs is the length of an image of the segment, the other leg is the difference of distances from the image plane.

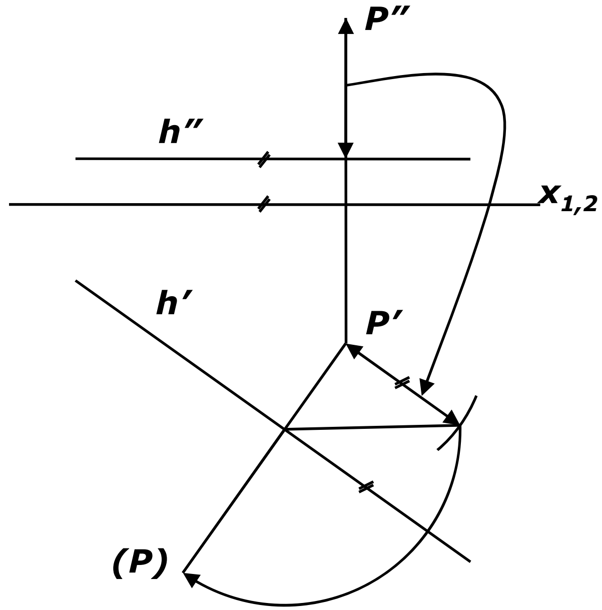


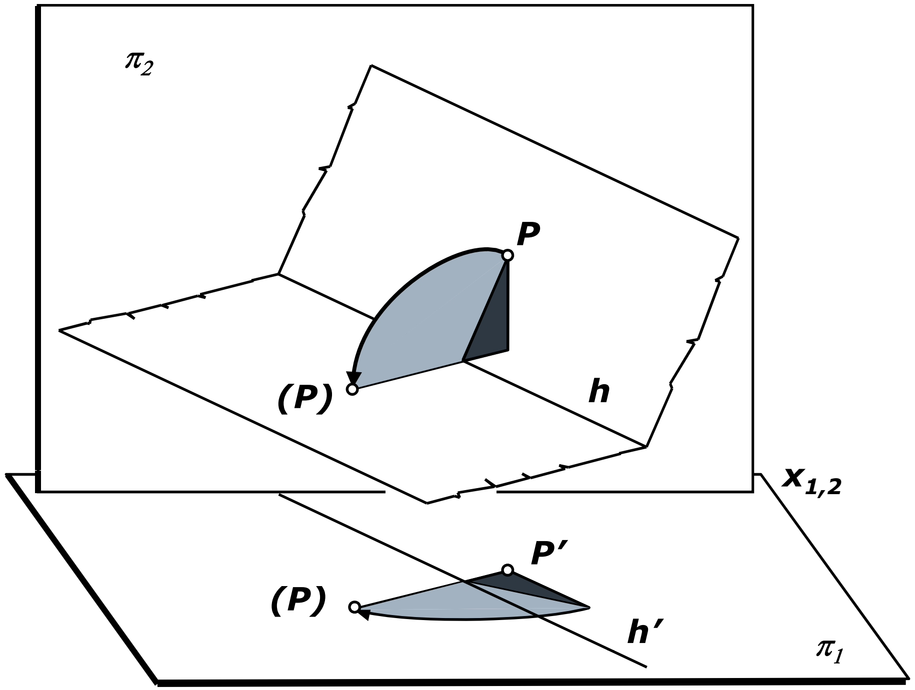


Reverse problem: the images of a line, a point of the line and a distance is given. Find the images of points of the line whose true distance from the given point is equal to the given distance. (Hint: by using an auxiliary point of the line find the ratio of the true length and the length of image.)

Basic Metrical Constructions 2

Any plan geometrical construction can be carried out by rotating the plane parallel to an image plane. The relation between the image of a plane and the image of the rotated plane is **orthogonal axial affinity**. The axis is a principal line of the plane. One rotated point can be found by the true distance of the point and the axis.



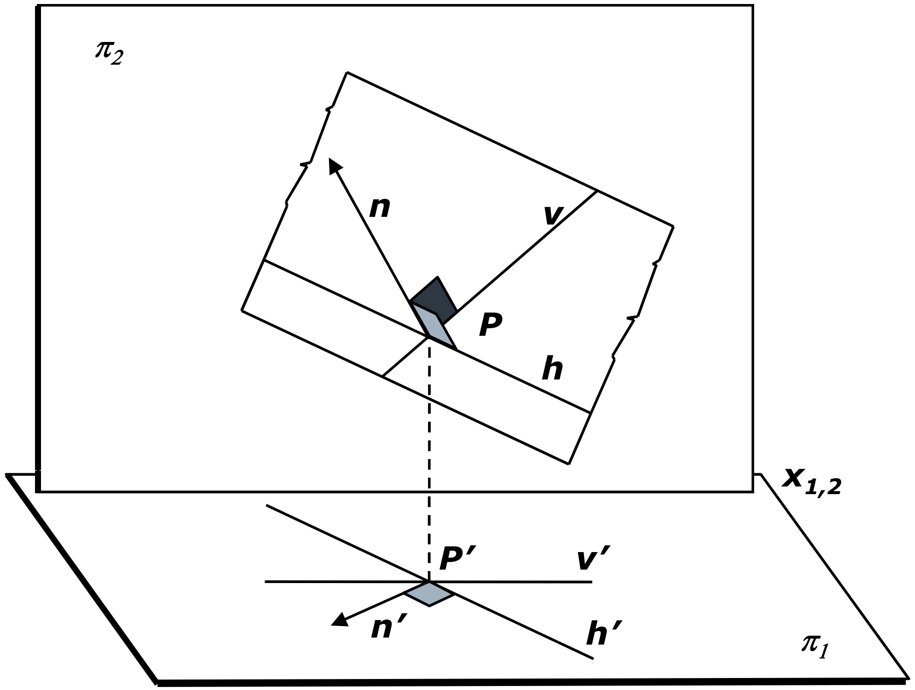


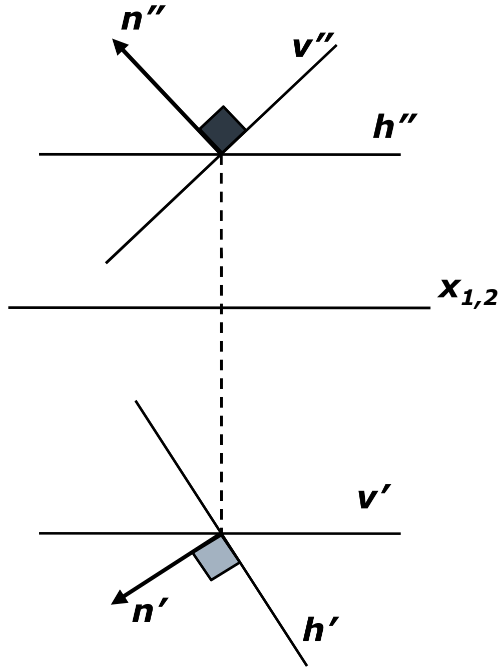
Reverse problem: construct the images of a figure, whose rotated image is given. Hint: use inverse affinity and lying on condition.

Basic Metrical Constructions 3

The first image of a normal of plane, ***n'*** is perpendicular to the first image of the first principal line ***h'*** of the plane.

The second image of a normal of plane, ***n''*** is perpendicular to the second image of the second principal line ***v''*** of the plane.



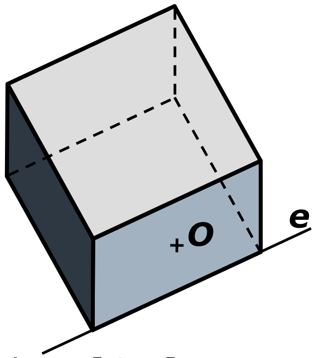


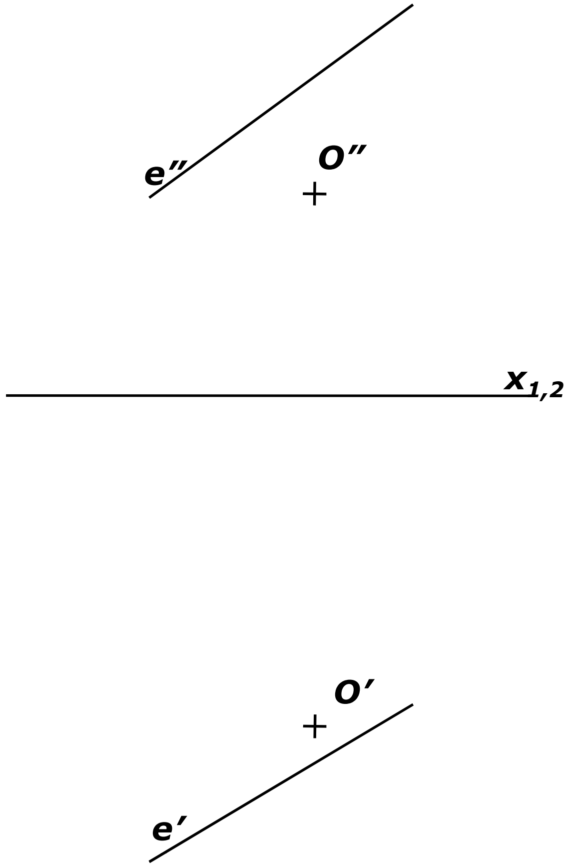
Reverse problem: construct a plane perpendicular to a given line.

Hint: the plane can be determined by means of principal lines.

Modeling of 3D Polyhedrons

Construct a cube. One of the faces is given by its center and line of an edge.



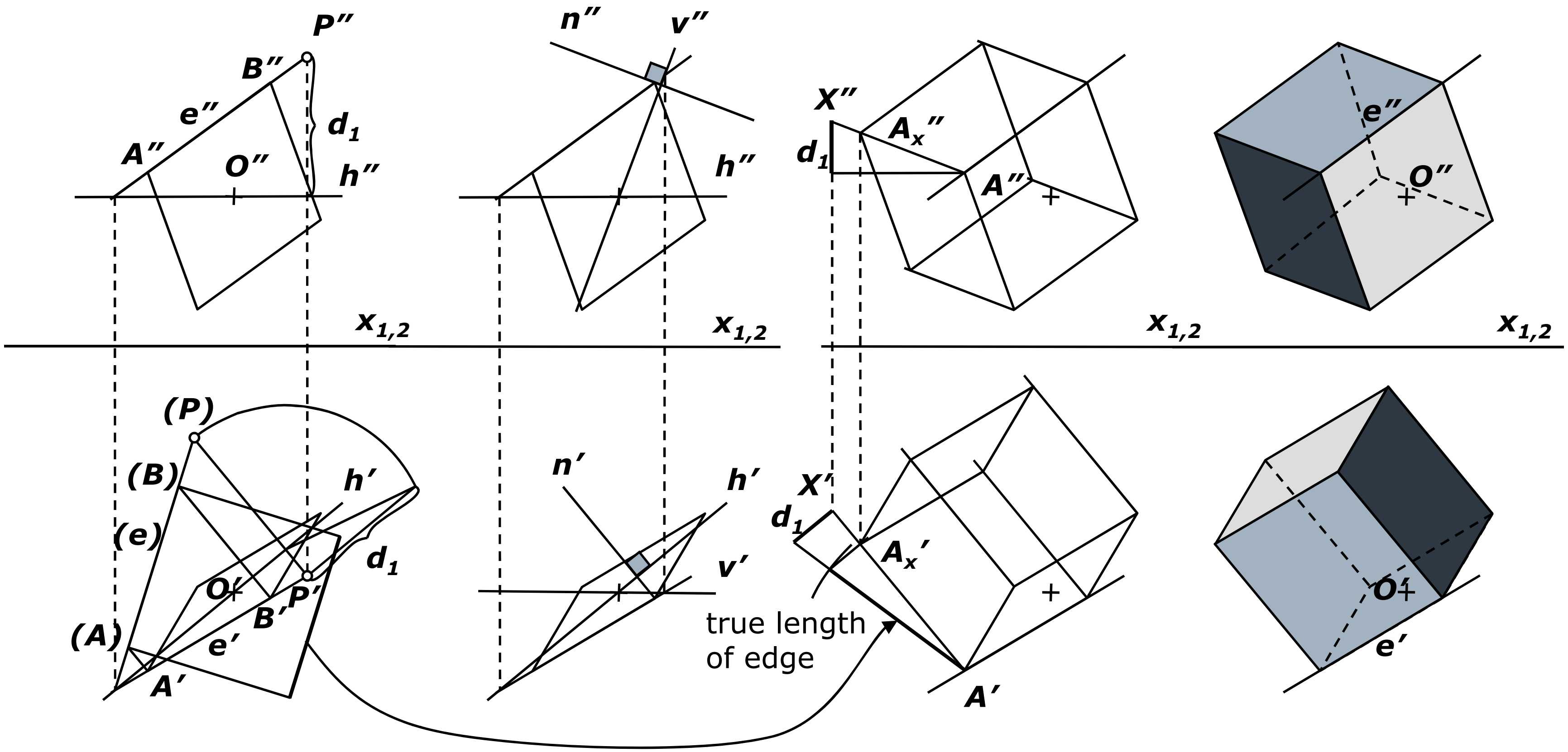


Algorithm

1) Construct the square lying in plane [***O,e***], with the centre ***O*** and an edge on ***e***. (Rotation - counter-rotation of plane, affinity, inverse affinity, (2).)
2) Construct lines perpendicular to the plane [***O,e***], passing through the vertices of the square. (Perpendicularity of line and plane, (3).)
3) Measure the length of an edge onto the perpendiculars, chose the proper direction from the two possibilities. (True length of a segment, (1).)
4) Complete the figure by showing the visibility.

Step-by-step Construction

Construction of the square | Normal of the plane | Measure of distance | Visibility



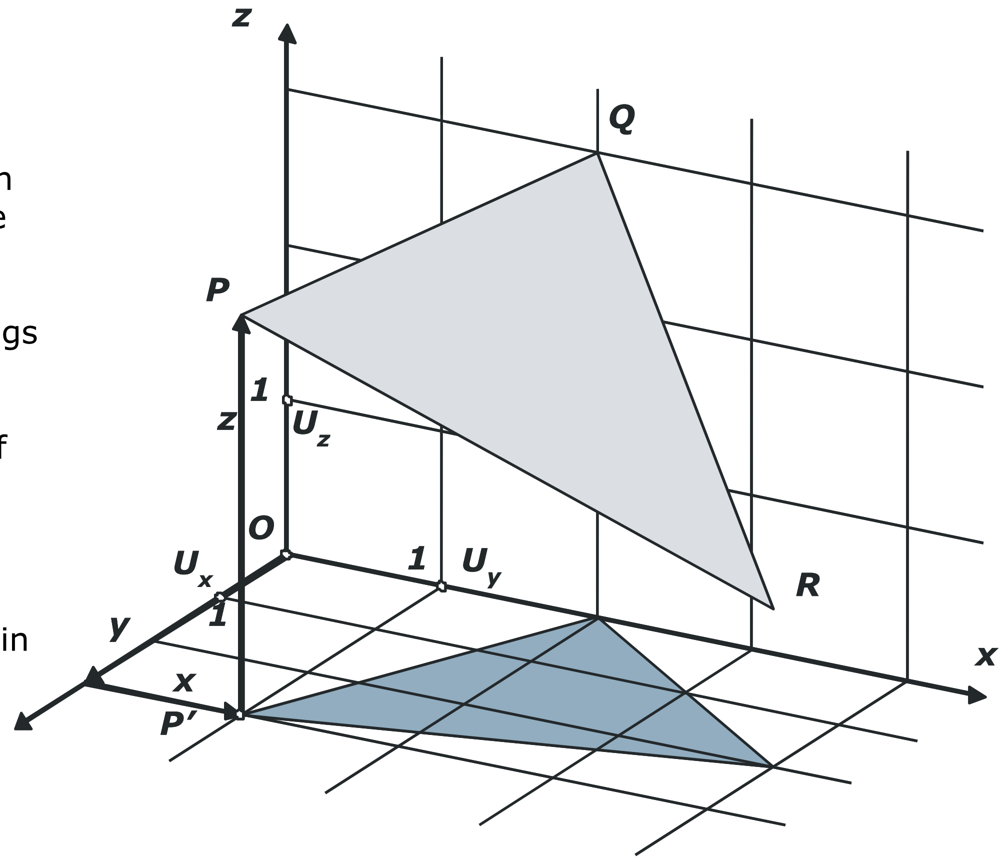
Axonometry

One of the methods of Descriptive Geometry, used to produce pictorial sketches for visualization.

Let three axes ***x***, ***y***, ***z*** through the origin ***O*** be given in the image plane. Measure the coordinates from ***O*** onto the three axes such that each coordinate will be multiplied by the ratios of foreshortenings *qx*, *qy*, *qz* respectively. The point determined by the coordinates is considered as the axonometric image of the point ***P***(*x*, *y*, *z*).

The axonometric system can be determined by the points {***O***, ***Ux***, ***Uy***, ***Uz***}, the image of the origin and the units on the axes ***x***, ***y*** and ***z***.

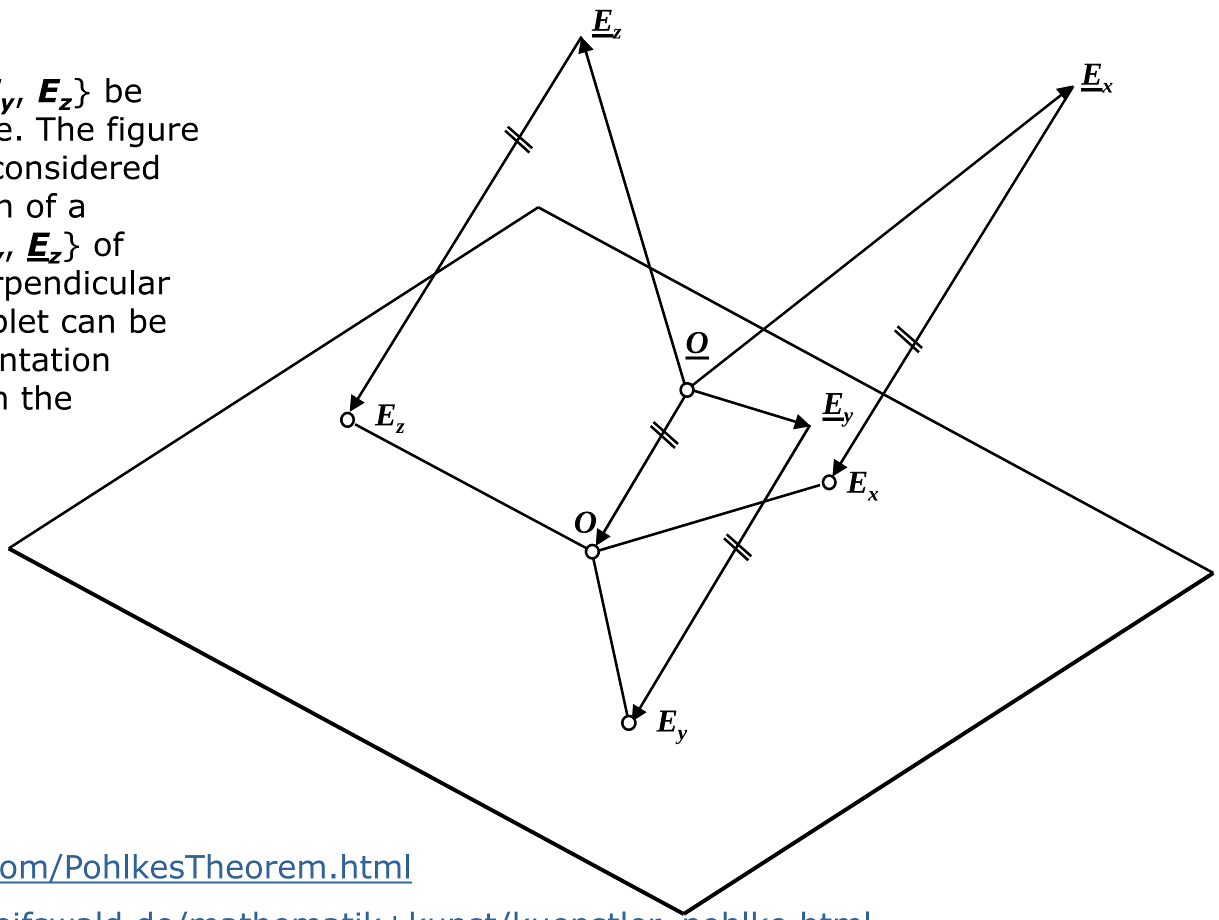
According to the Fundamental Theorem of Axonometry the axonometric image of an object is a parallel projection or similar to the parallel projection of the object.



In axonometry the left-handed Cartesian system is used.

Fundamental Theorem of Axonometry (Pohlke)

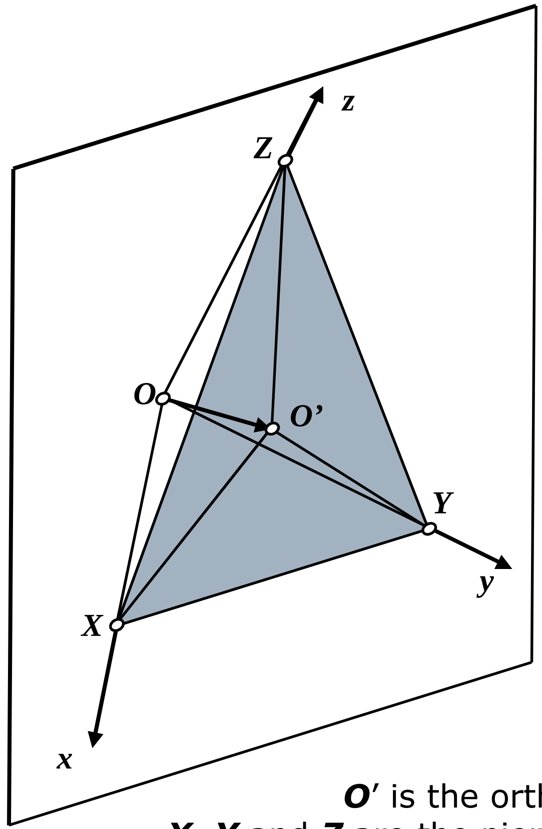
Let the sytem $\{\boldsymbol{O}, \boldsymbol{E_x}, \boldsymbol{E_y}, \boldsymbol{E_z}\}$ be given in the image plane. The figure $\{\boldsymbol{O}, \boldsymbol{E_x}, \boldsymbol{E_y}, \boldsymbol{E_z}\}$ can be considered as the parallel projection of a spatial triplet $\{\underline{\boldsymbol{O}}, \underline{\boldsymbol{E}}_{\boldsymbol{x}}, \underline{\boldsymbol{E}}_{\boldsymbol{y}}, \underline{\boldsymbol{E}}_{\boldsymbol{z}}\}$ of three unit segments perpendicular by pairs. The spatial triplet can be one of two types of orientation apart from translation in the direction of projection.

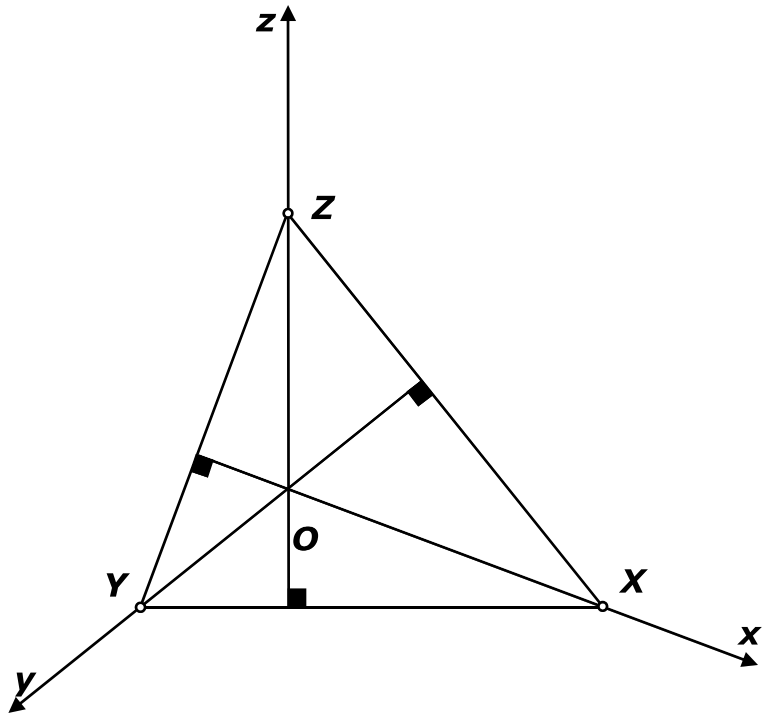


http://mathworld.wolfram.com/PohlkesTheorem.html

http://www.math-inf.uni-greifswald.de/mathematik+kunst/kuenstler_pohlke.html

Orthogonal Axonomtry



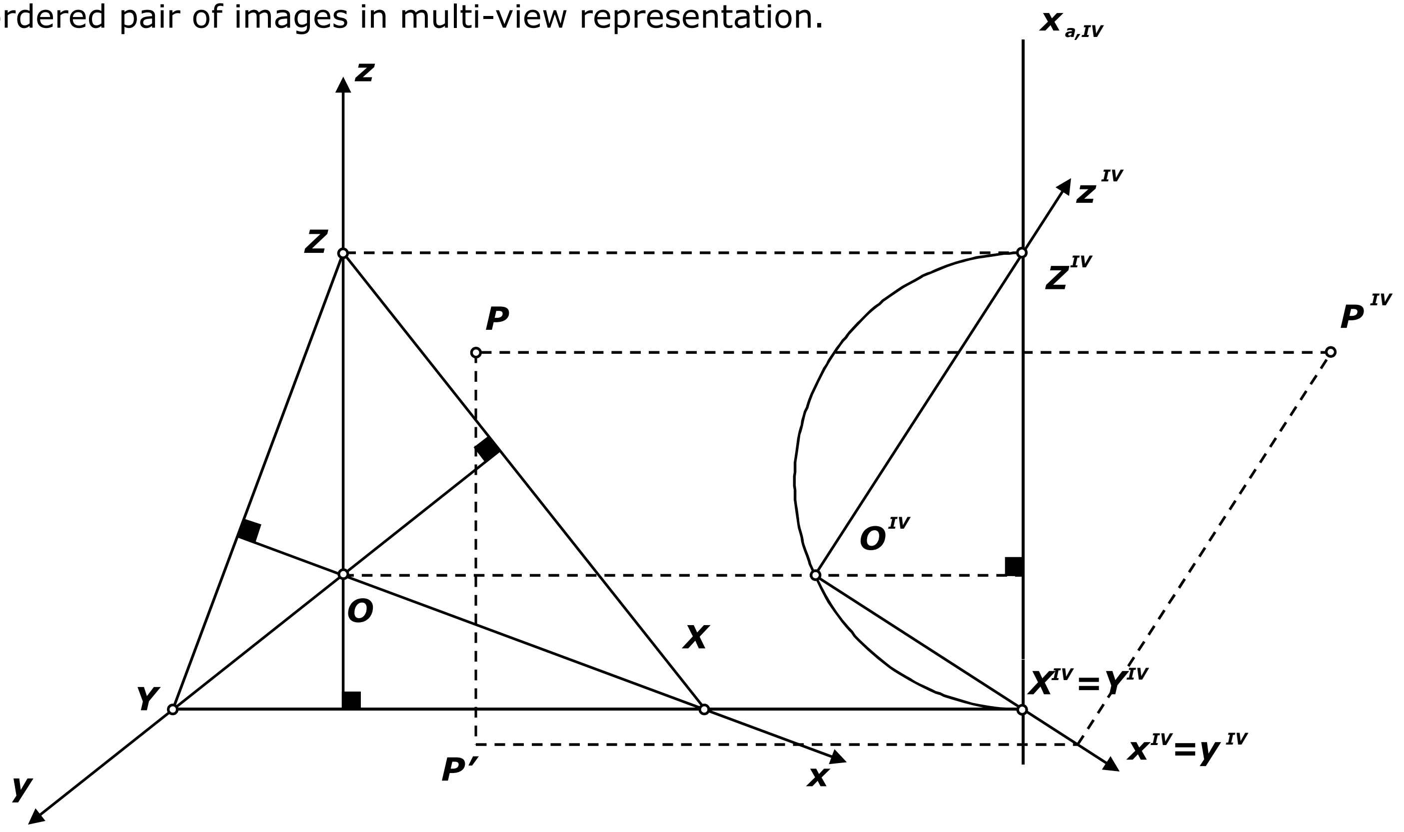


O' is the orthogonal projection of the origin,
X, ***Y*** and ***Z*** are the piercing points of the axes in the image plane.
O' is the orthocenter of the triangle (tracing triangle) ***XYZ***.

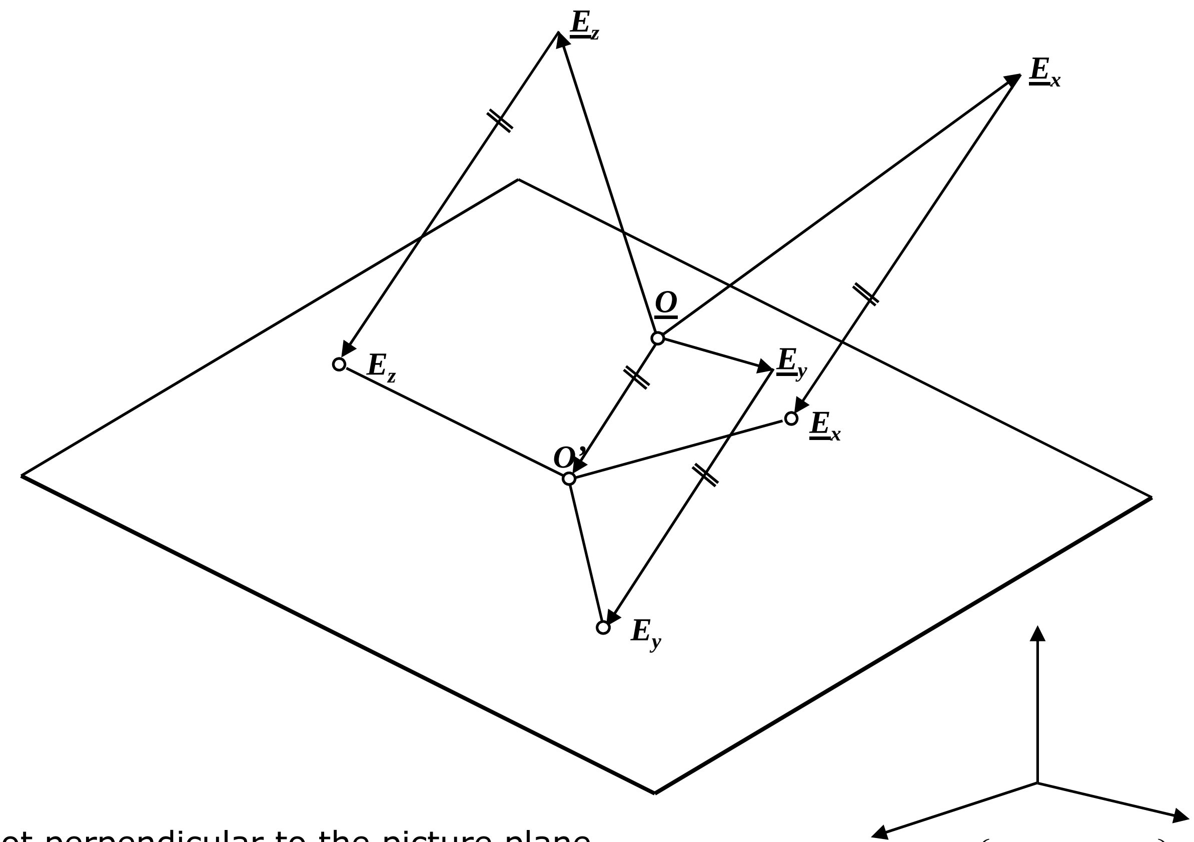
In the axonometric sketch the prime (') is omitted.

Orthogonal Axonometry → Multi-view

The orthogonal axonometric image can be considered as one of an ordered pair of images in multi-view representation.



Oblique (klinogonal) Axonometry

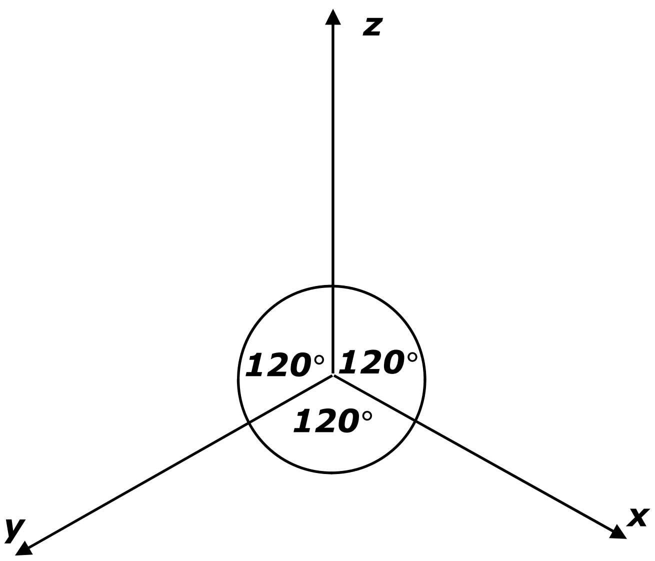


OO' <u>is not perpendicular</u> to the picture plane.

$\{q_x, q_y, q_z\}$

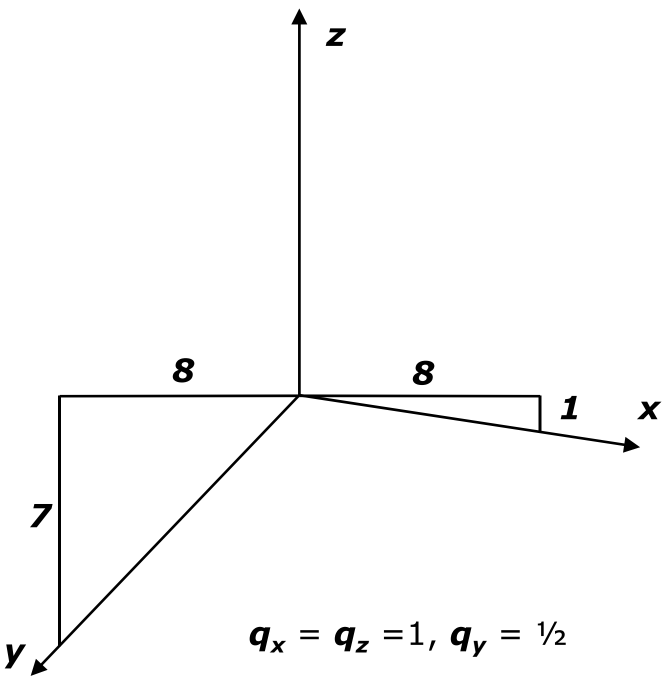
Izometry, Technical Axonometry

Izometry



$q_x = q_y = q_z = 1$

Technical axonometry



$q_x = q_z = 1,\ q_y = ½$

Cavalier, Bird’s-eye View, Worm’s-eye View

Frontal Axonometry

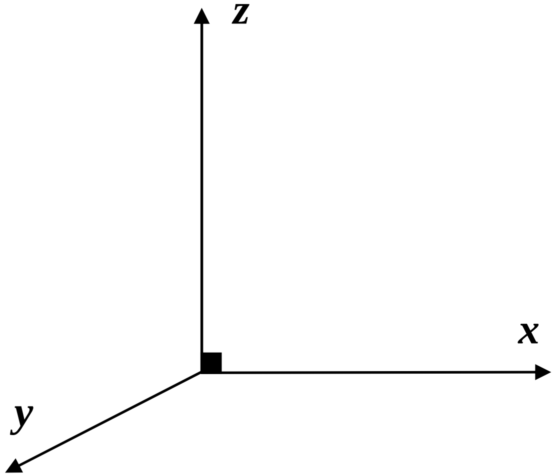


Image plane: [***xz***]

if $\boldsymbol{q_y} = 1$: **cavalier axonometry**

Bird’s eye view (top view)
Military axonometry

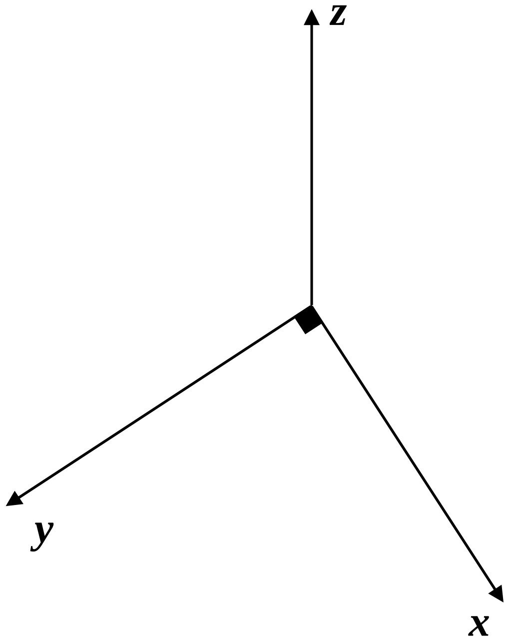


Image plane: [***xy***] $\boldsymbol{q_x} = \boldsymbol{q_y} = 1$

if $\boldsymbol{q_z} = 1$: **military axonometry**

Worm’s eye view (bottom view)

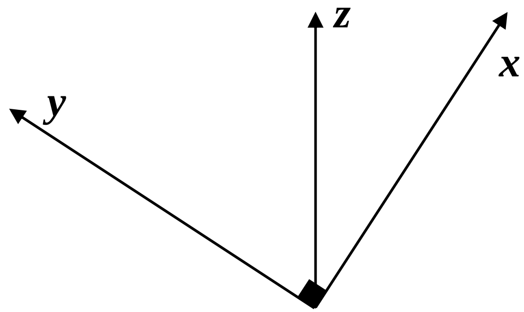
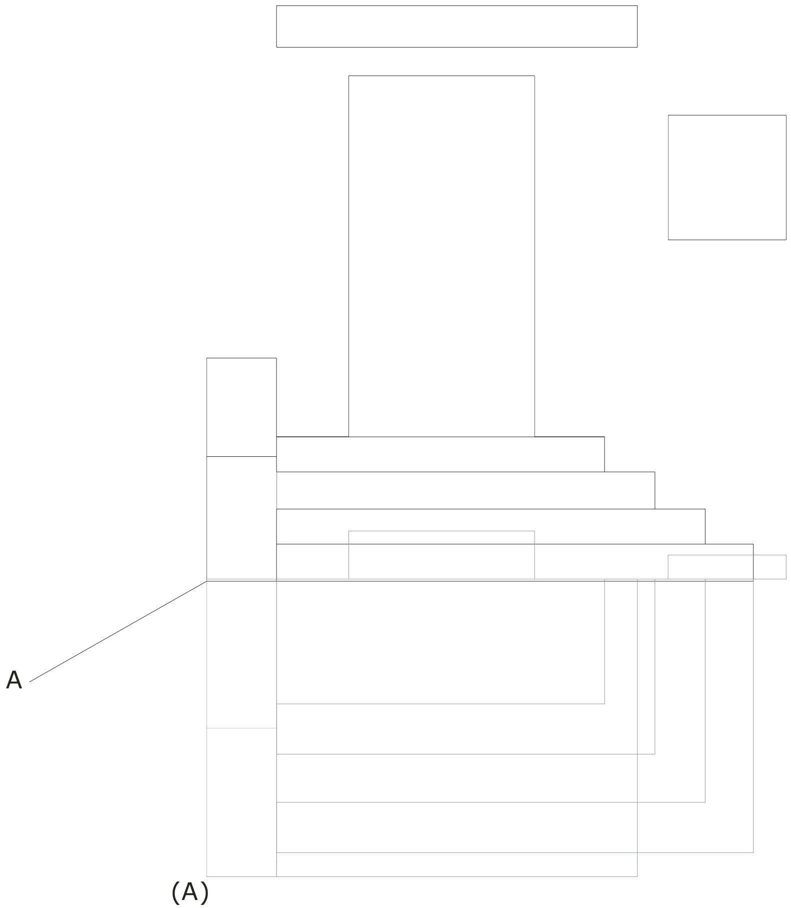
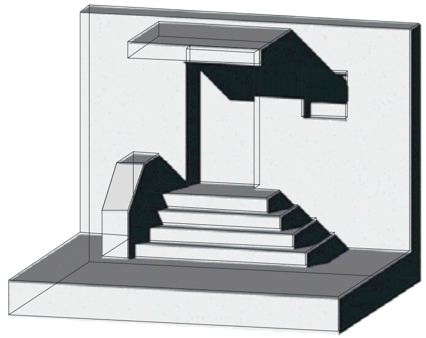
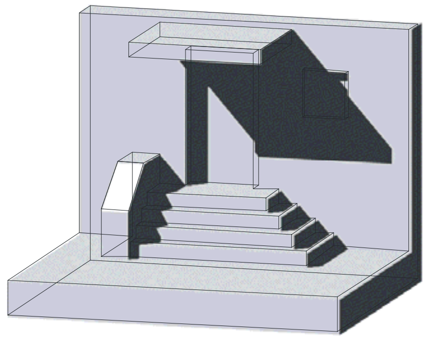


Image plane: [***xy***]
$\boldsymbol{q_x} = \boldsymbol{q_y} = 1$

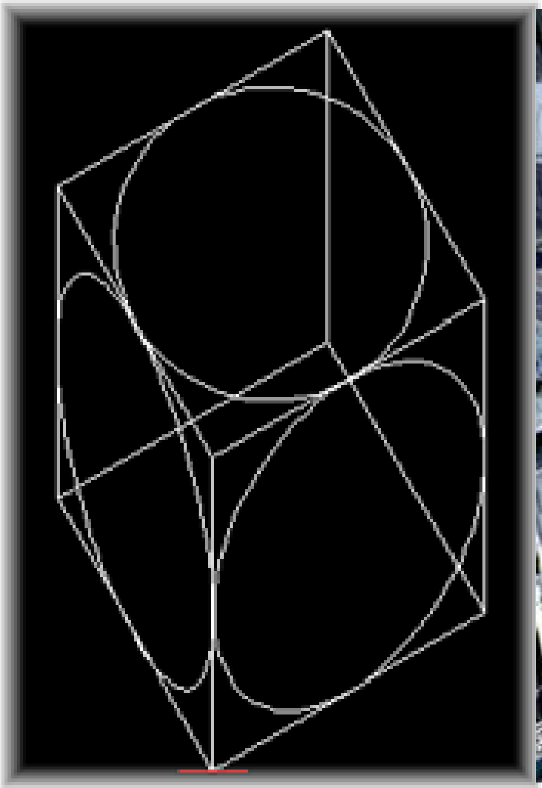
Frontal Axonometry, Shadow

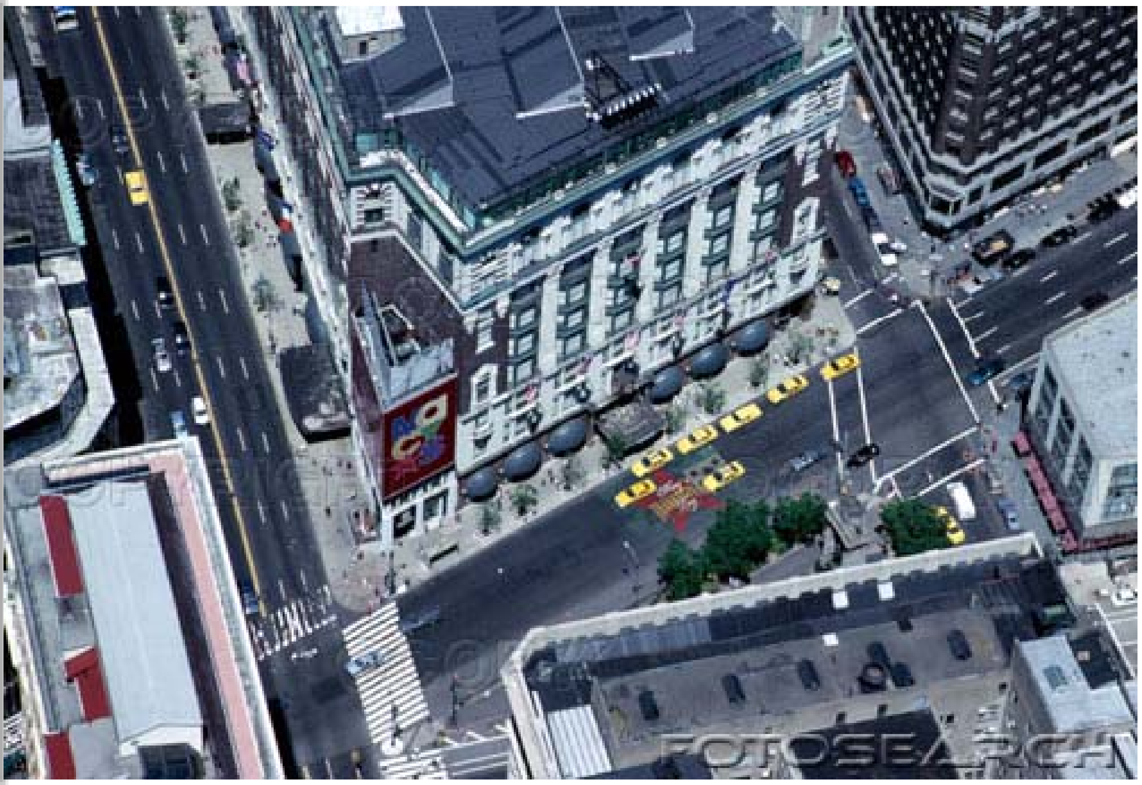






Military Axonometry

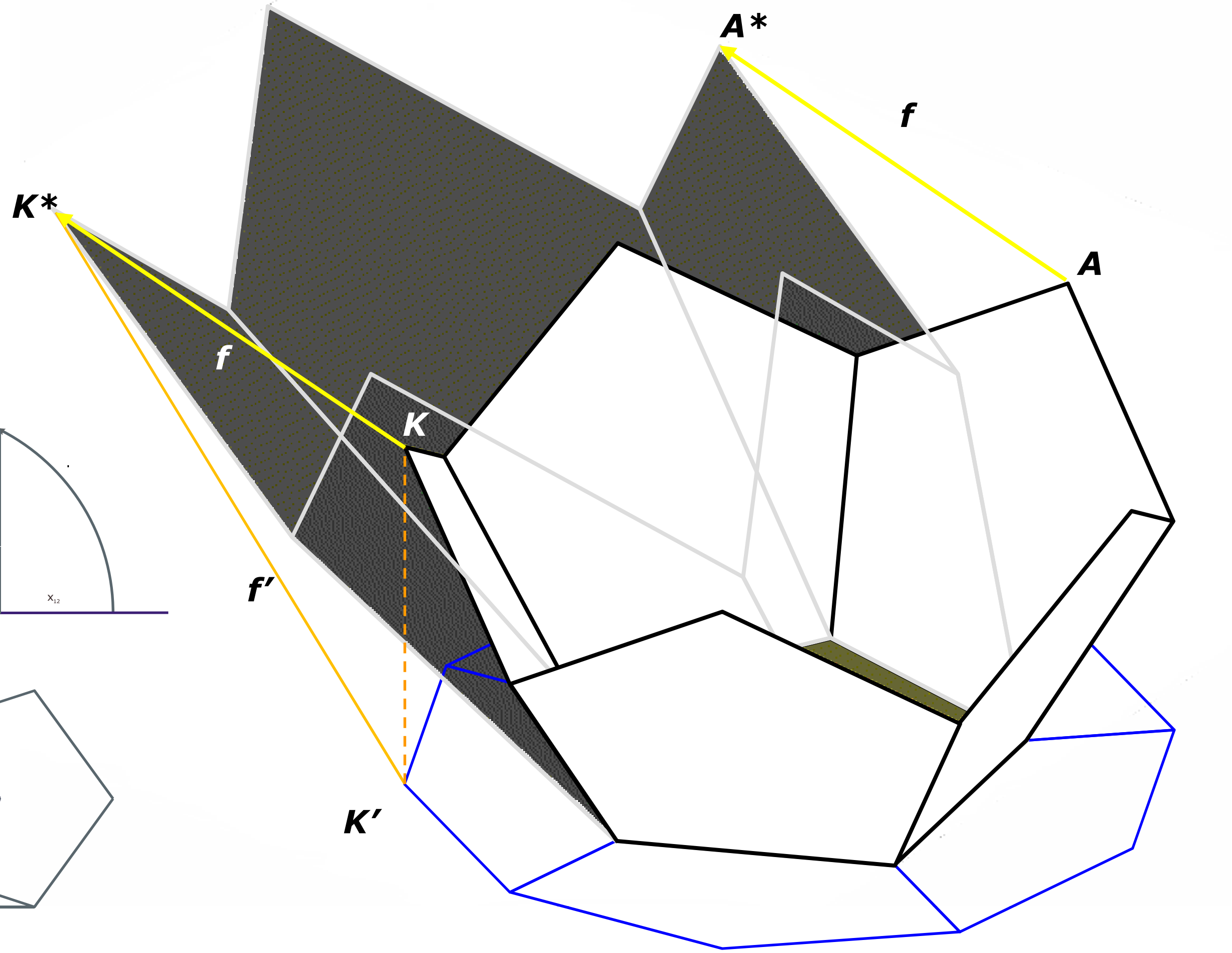




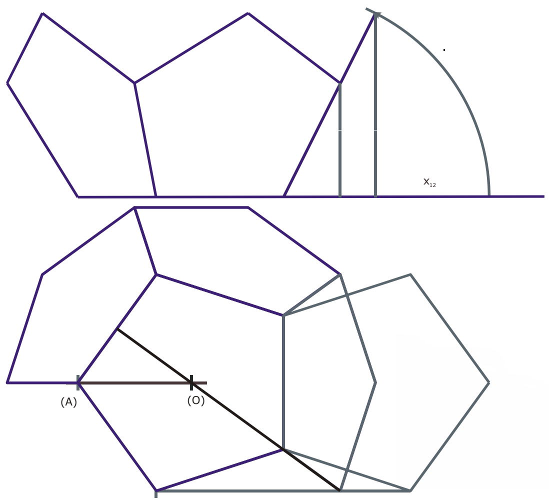
www.xanadu.cz

http://www.fotosearch.com/NGF001/57478808/

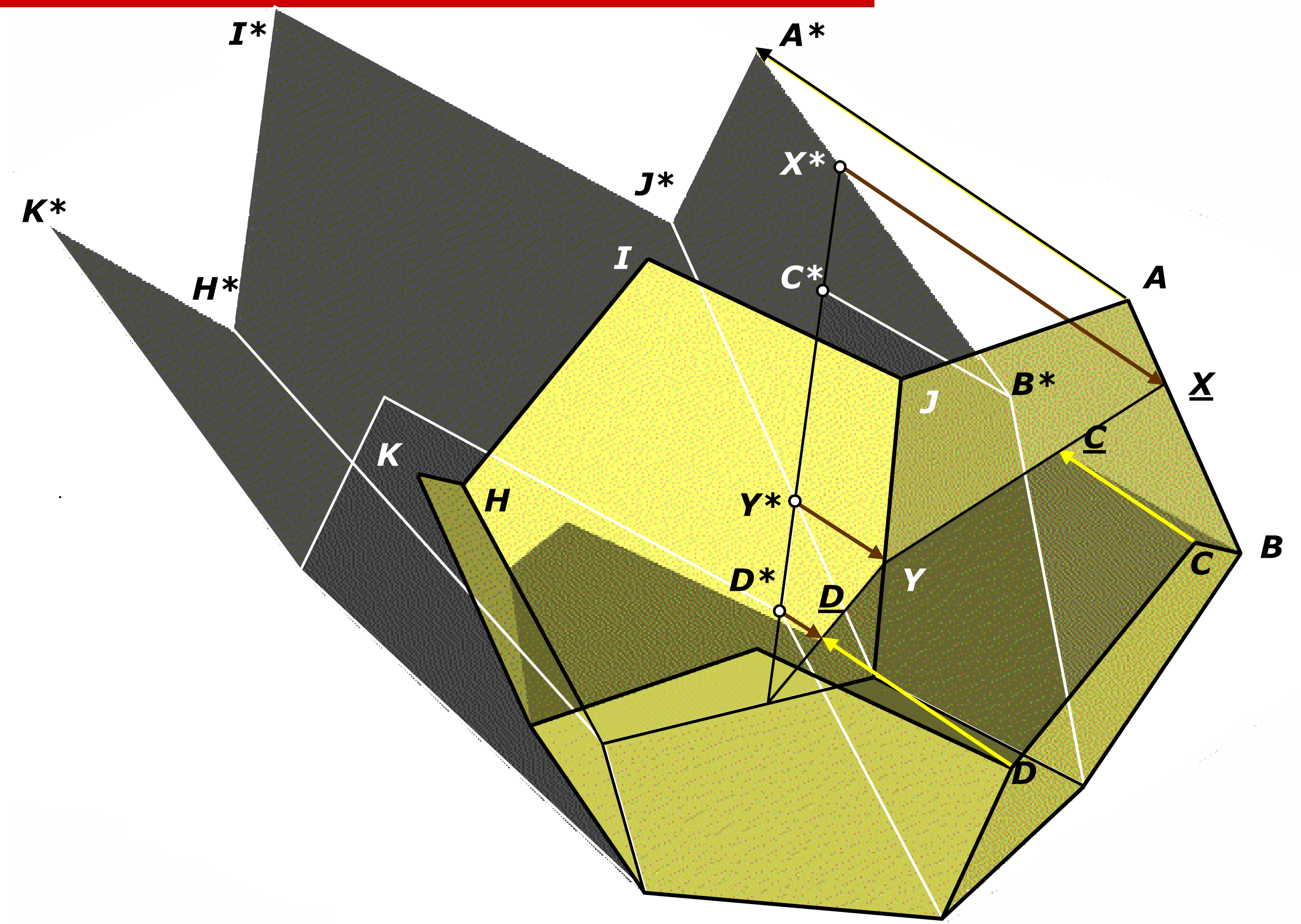
Cast Shadow in Orthogonal Axonometry



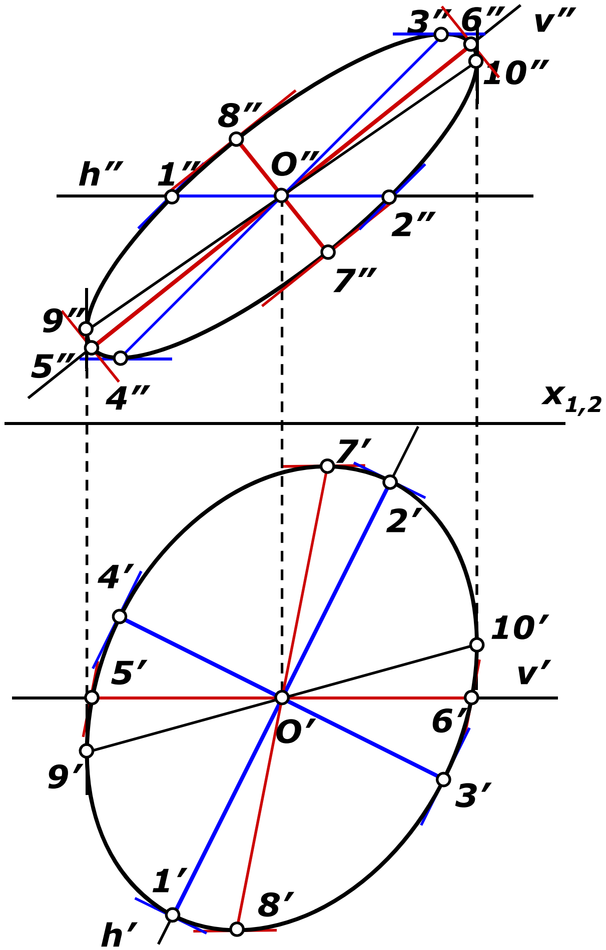
Dodecahedron,
top view and heights



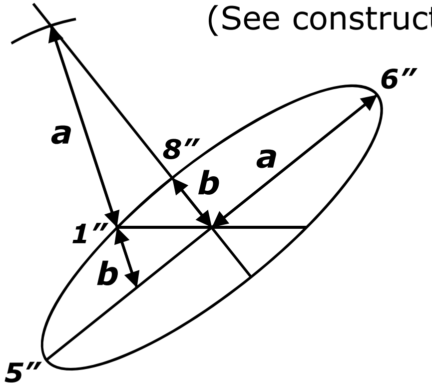
Projected Shadow



Representation of Circle (Multi-view)

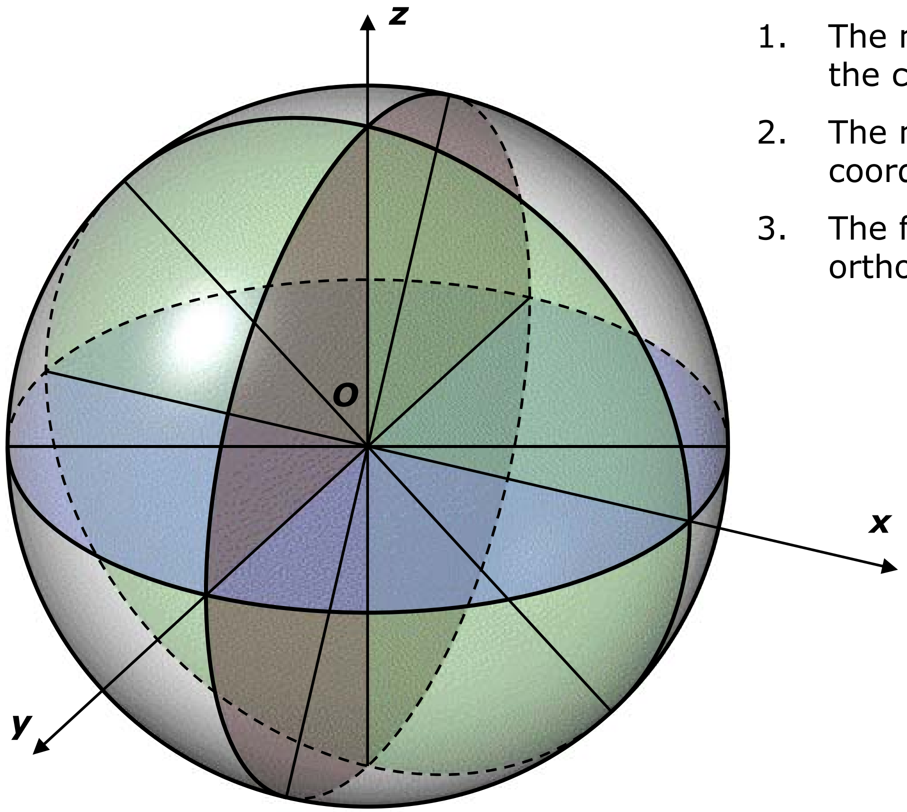


1. The major axes lie on first and second principal lines ***h′*** and ***v″*** respectively.
2. The length of major axes ***1′2′*** and ***5″6″*** is equal to the diameter of circle (true length).
3. The length of a minor axis is constructible from the major axis and a point, as plane geometric construction. (See construction of ***8″***)

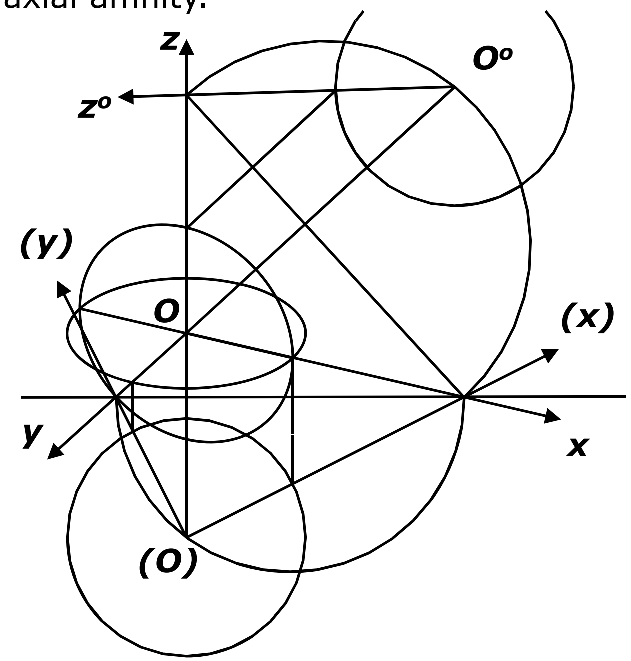


4. The left and right extreme points ***9*** and ***10*** can be found as points of ellipse with vertical tangents, by means of orthogonal axial affinity.
5. The tangents at the points mentioned above are parallel to the proper diameters.

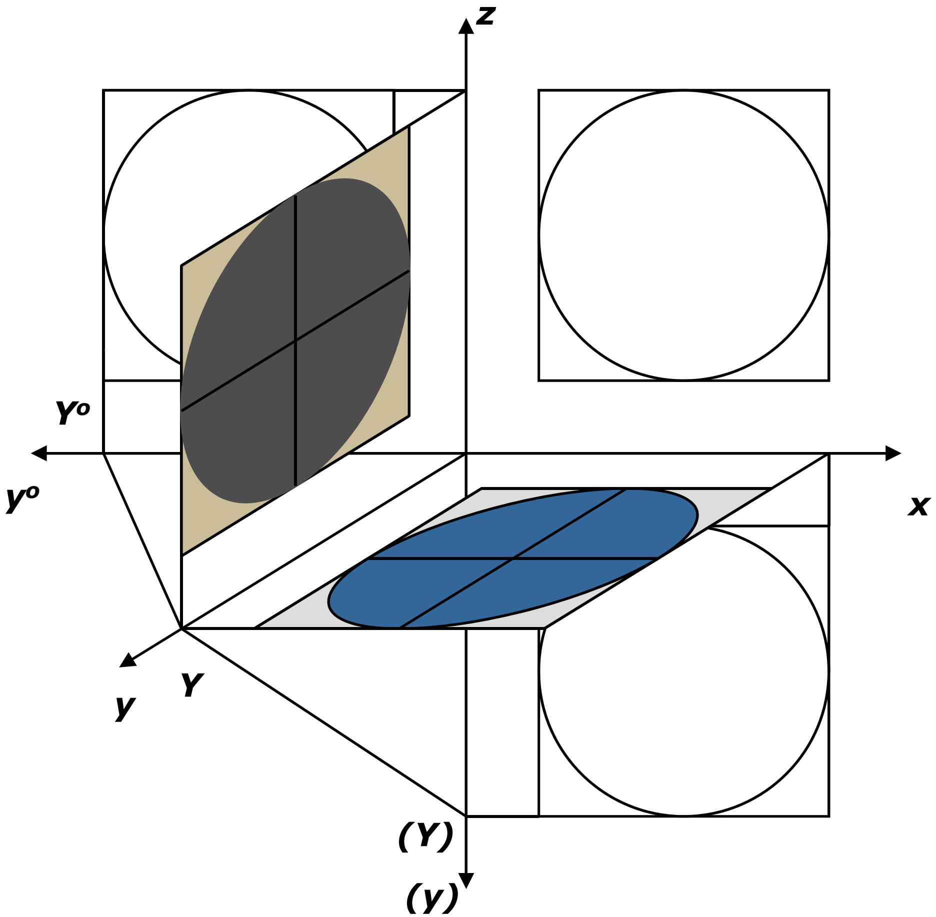
Representation of Circle (Orthogonal Axonometry)



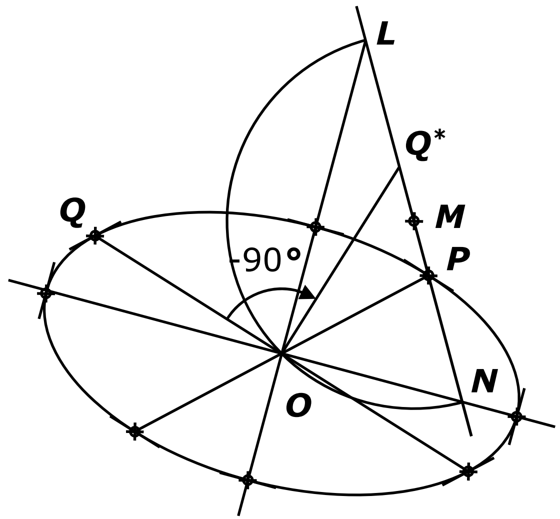
1. The major axes of ellipses are perpendicular to the coordinate axes.
2. The minor axes are coinciding lines with the coordinate axes.
3. The fundamental method of constructions is the orthogonal axial affinity.



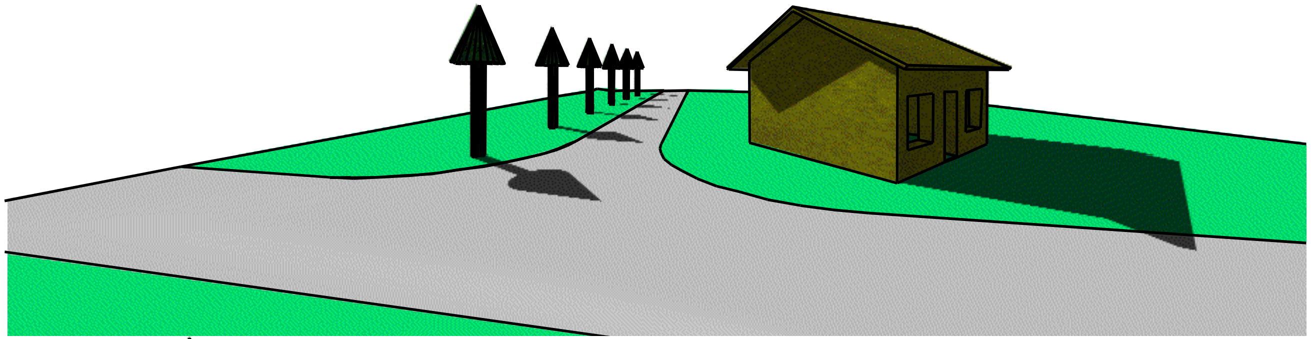
Representation of Circle (Oblique Axonometry)

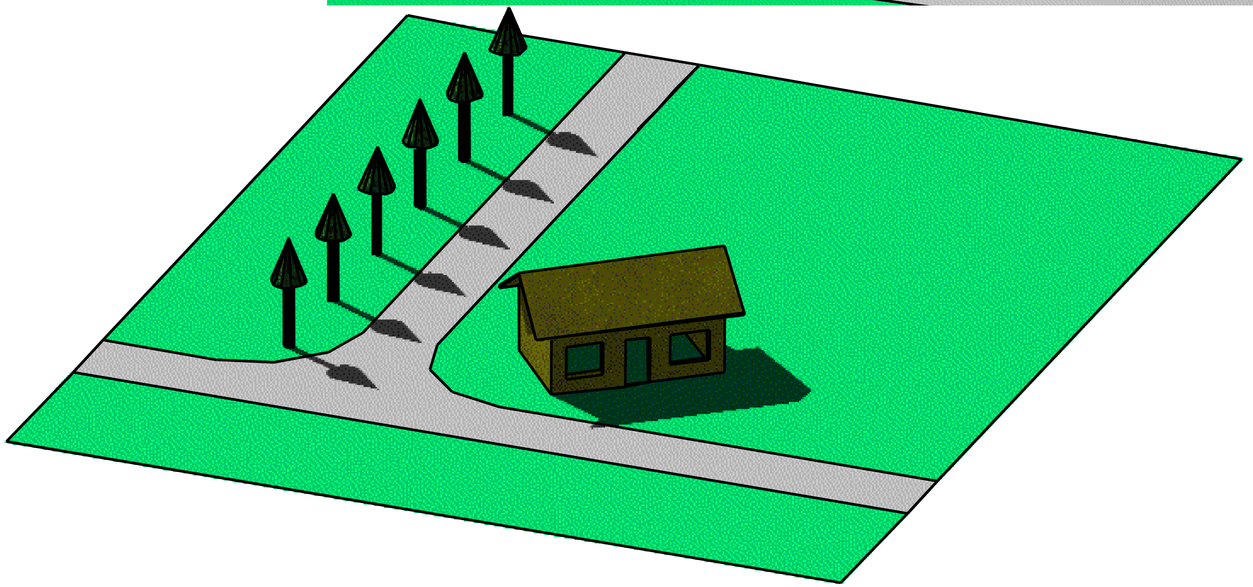


1. The fundamental method of constructions is the oblique axial affinity.
2. The axes can be constructed by Rytz' method. The ellipse is determined by a pair of conjugated diameters; find the major and minor axes.

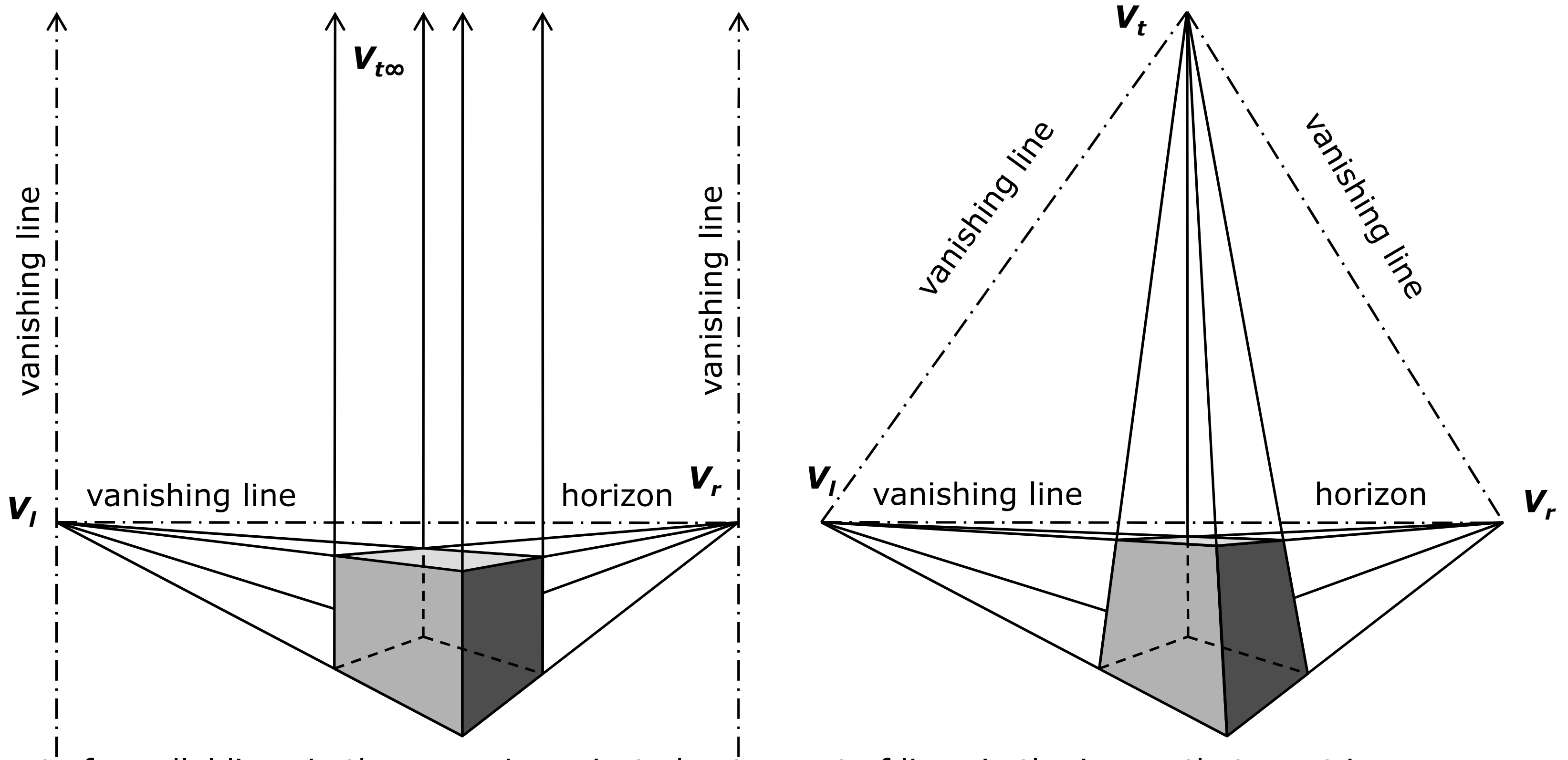


Axonometry vs. Perspective



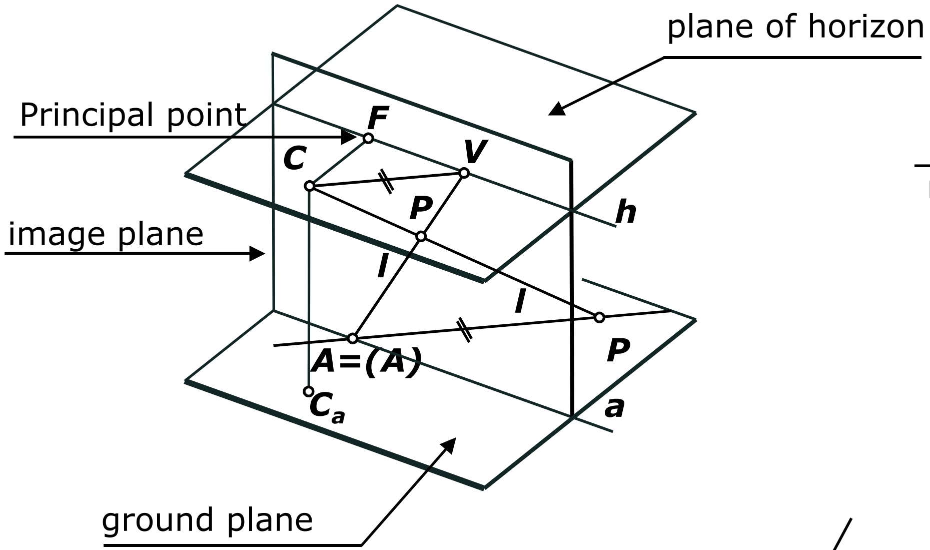


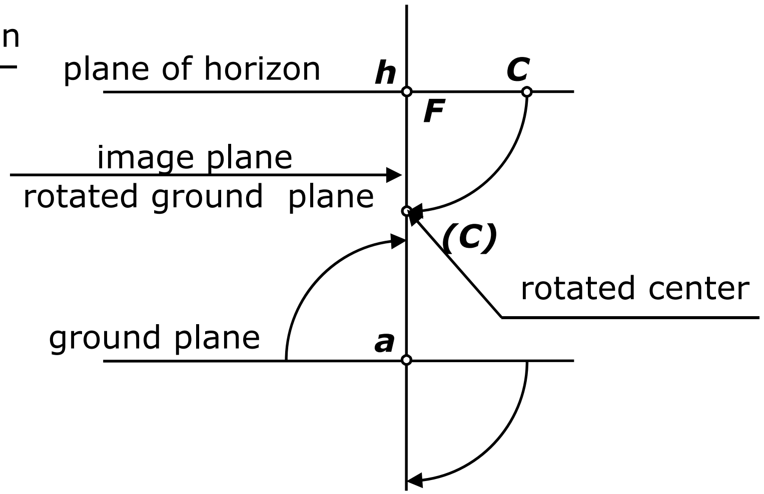
Vanishing Point, Vanishing Line

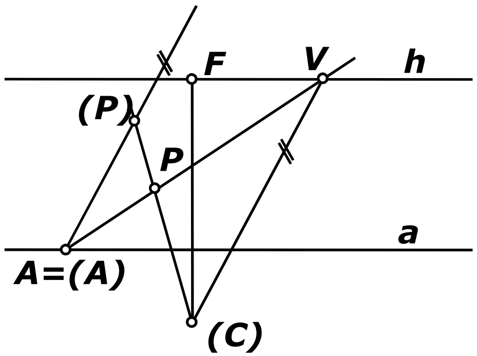


A set of parallel lines in the scene is projected onto a set of lines in the image that meet in a common point. This point of intersection is called the **vanishing point**. A vanishing point can be a finite (real) point or an infinite (ideal) point on the image plane. Vanishing points which lie on the same plane in the scene define a line in the image, the so-called the **vanishing line**.

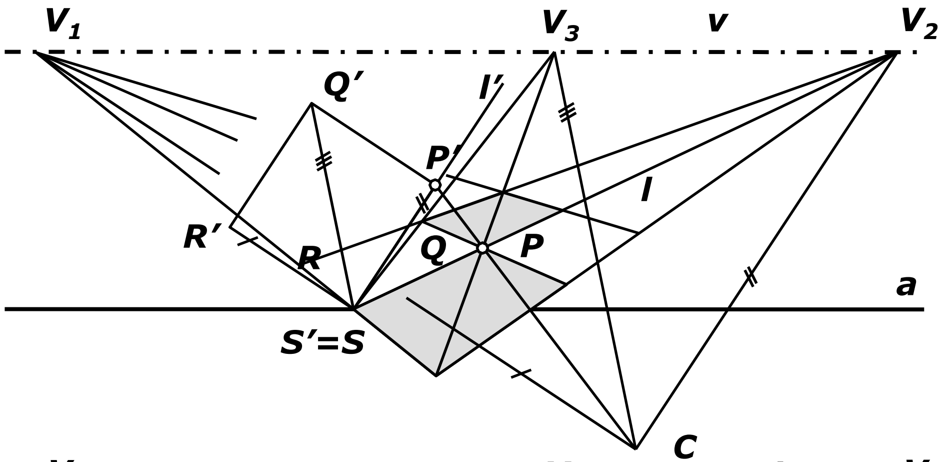
Basics of Perspective with Vertical Image Plane

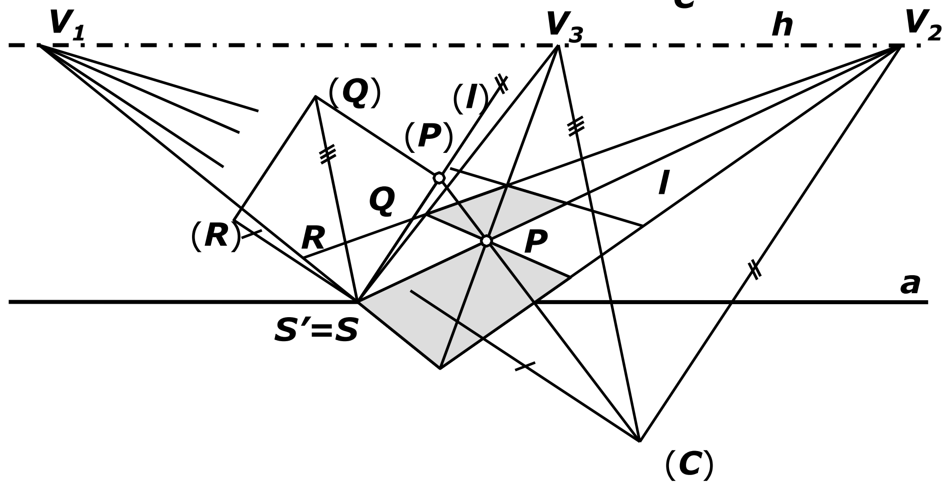






Perspective Collineation, Perspective Mapping





A perspective collineation is determined by the center ***C***, axis ***a*** and the vanishing line ***v***.

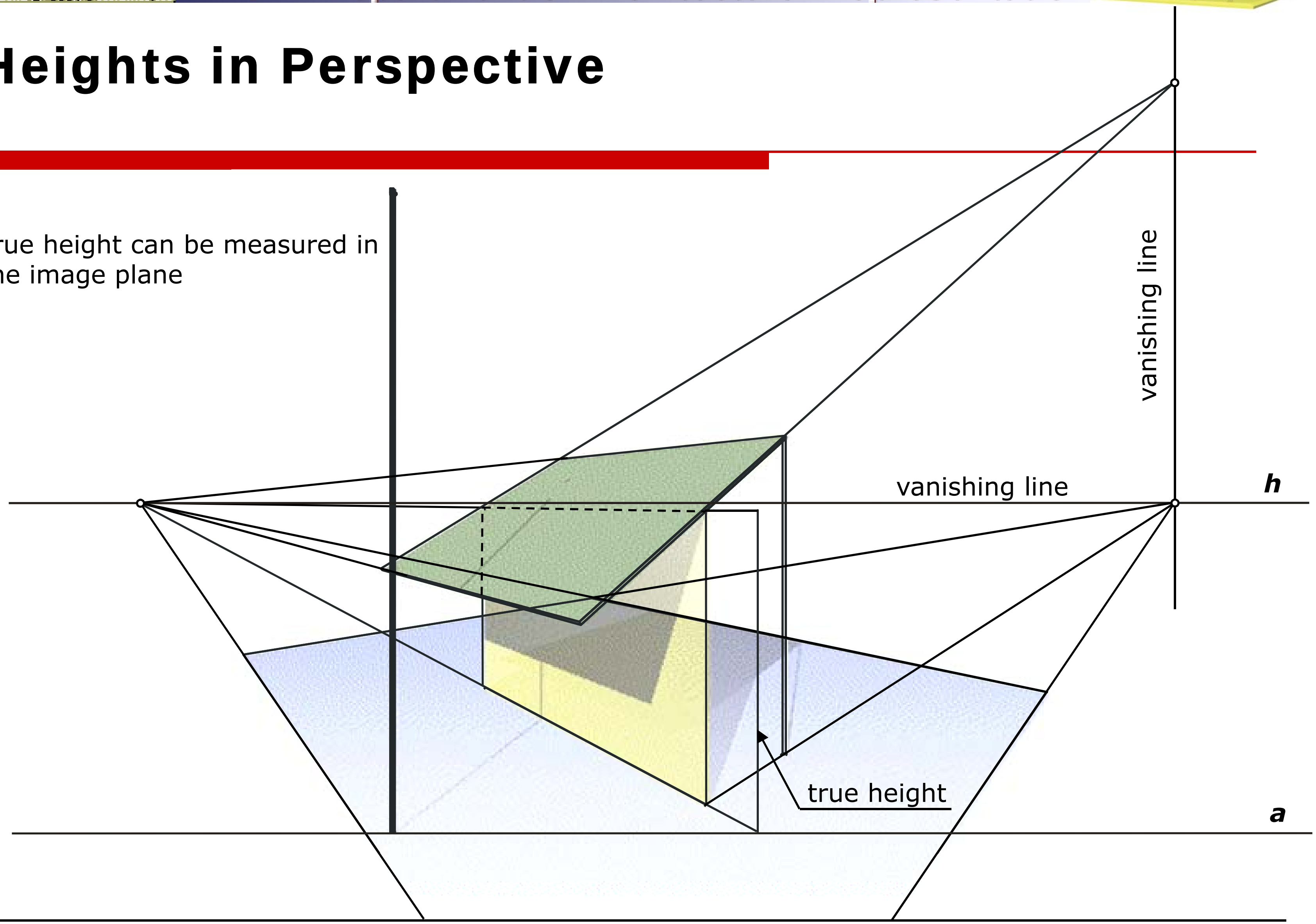
To the square ***P′, Q′, R′, S′=S***, we can find the quadrilateral ***PQRS*** at the mapping ***Π′*** ⇒ ***Π***.

When the ground plane is rotated into the picture plane, the two systems of points and lines are related by central-axial collineation. This perspective collineation is determined by the rotated center (***C***), axis ***a*** and the horizon ***h*** (vanishing line).

To the square (***P***), (***Q***), (***R***), (***S***)=***S***, we can find the quadrilateral ***PQRS*** at the mapping (***Π***) ⇒ ***Π***.

Heights in Perspective

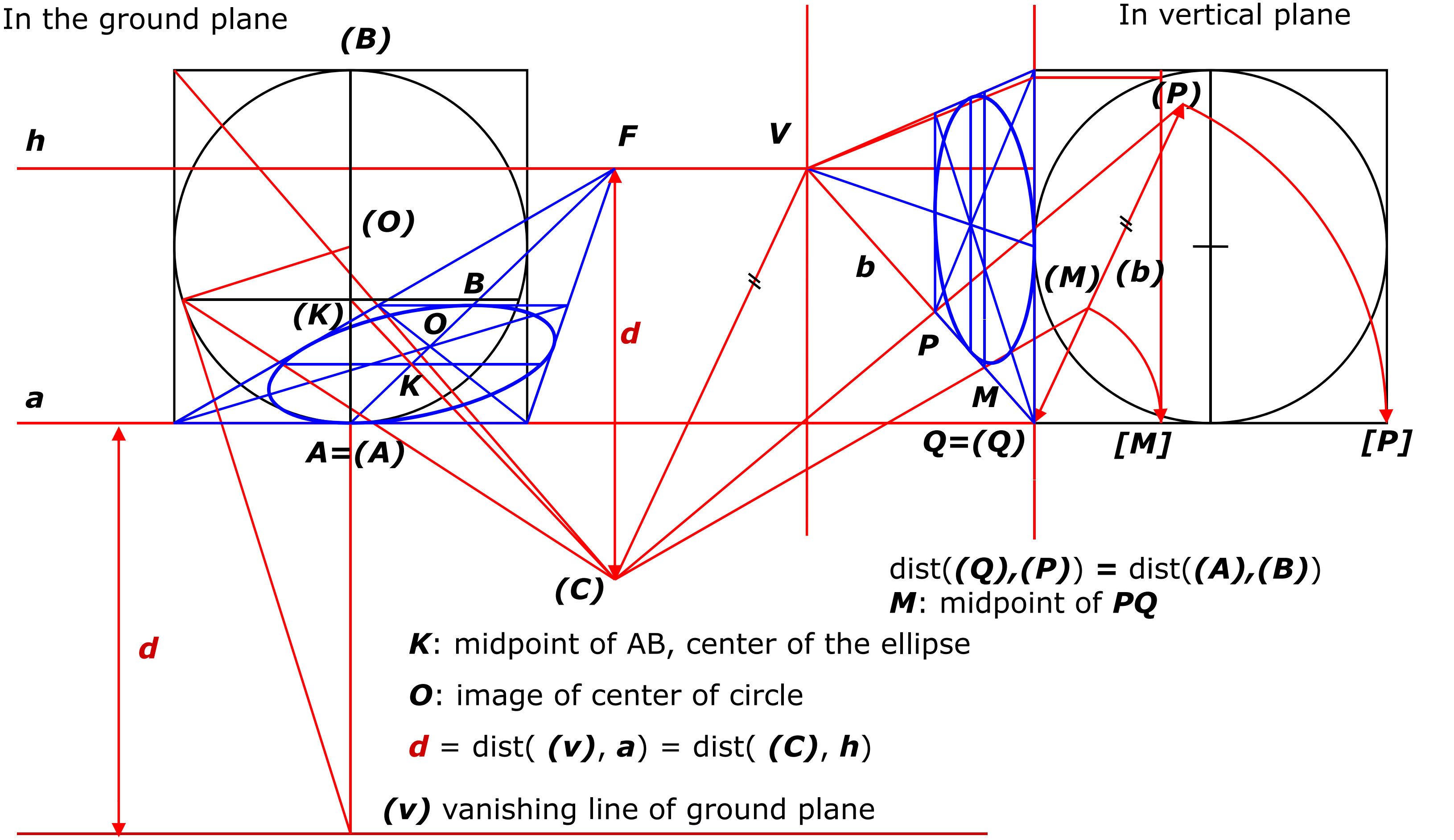
True height can be measured in the image plane



Representation of Circle (Perspective)

In the ground plane

In vertical plane



dist(***(Q),(P)***) = dist(***(A),(B)***)

M: midpoint of ***PQ***

K: midpoint of AB, center of the ellipse

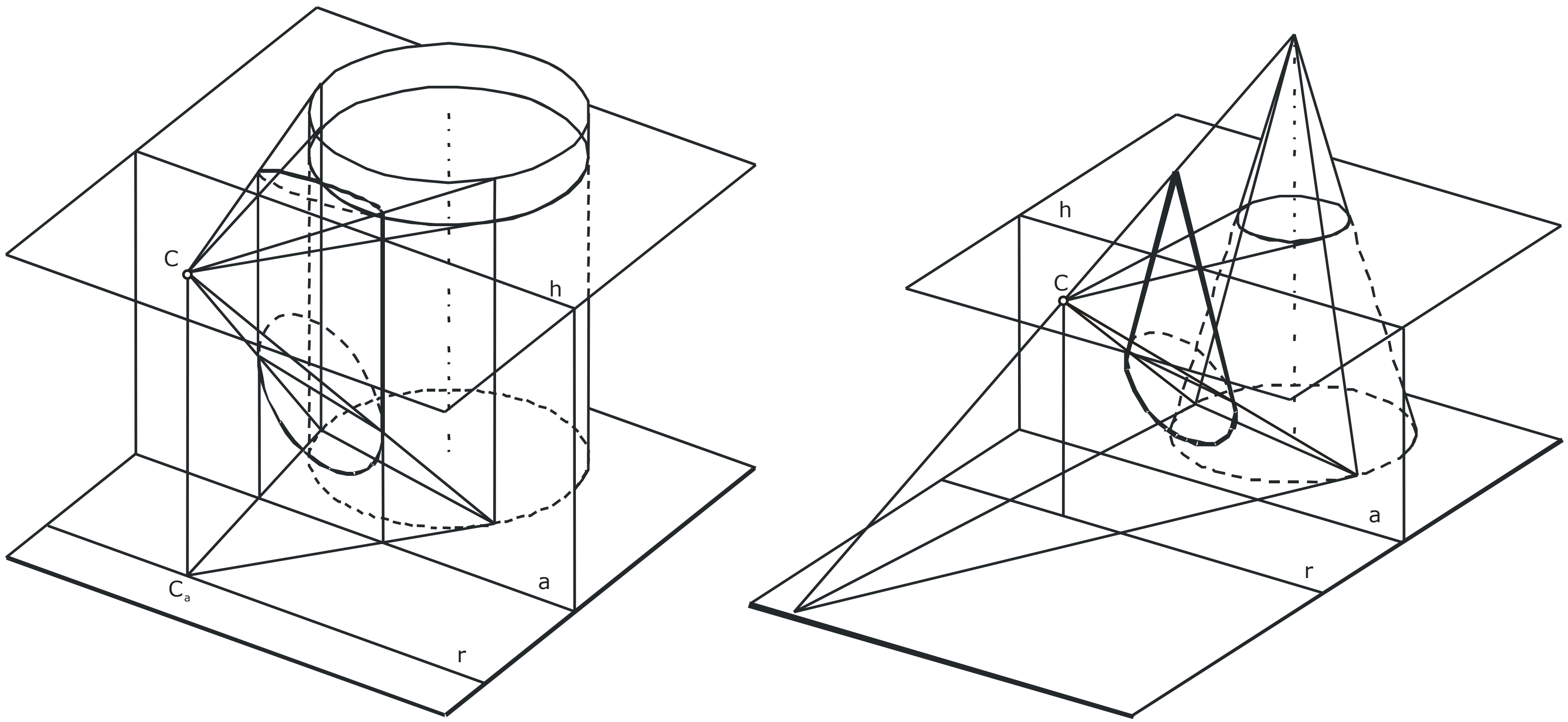
O: image of center of circle

d = dist(***(v)***, ***a***) = dist(***(C)***, ***h***)

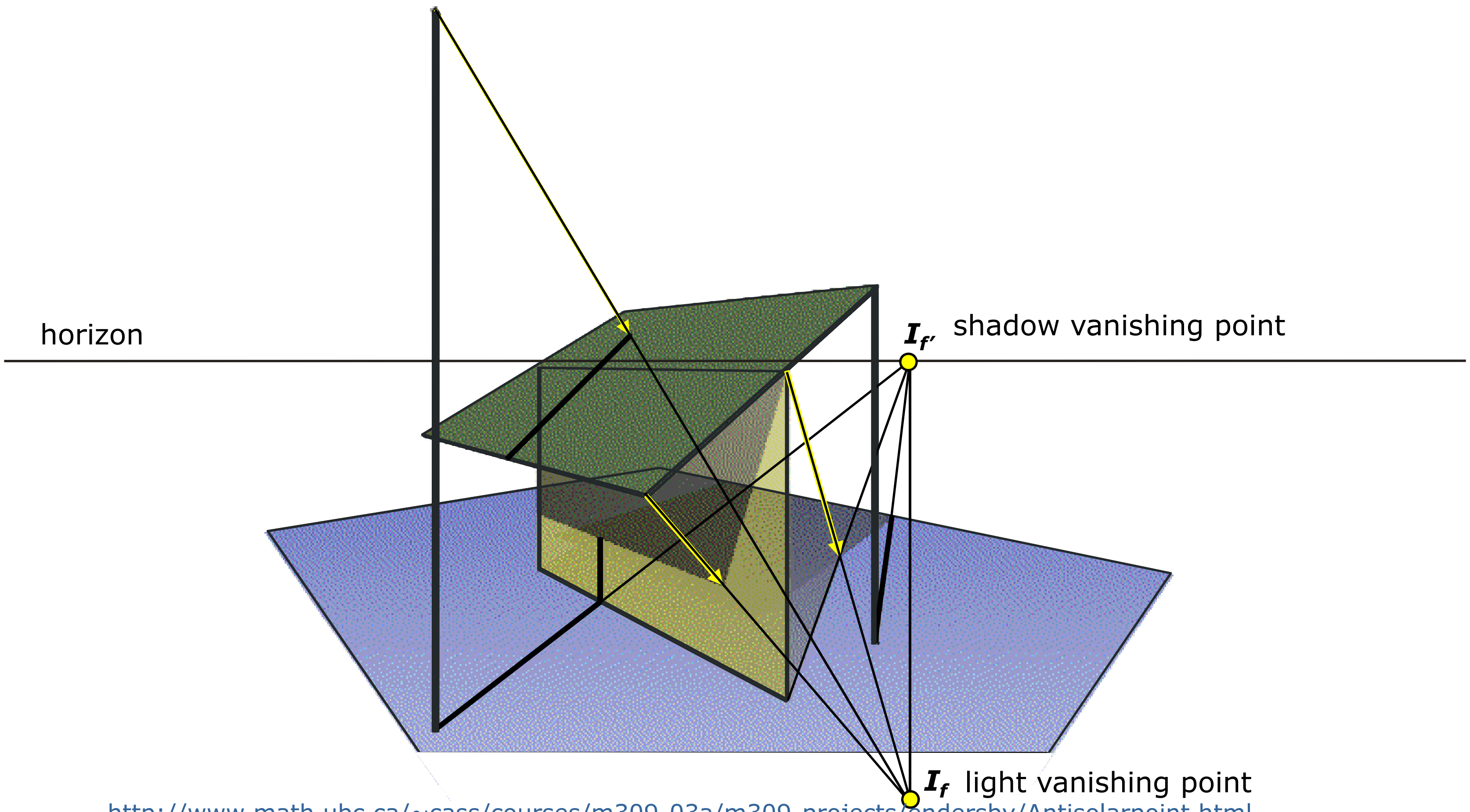
(v) vanishing line of ground plane

Cylinder and Cone in Perspective

Axonometric sketch

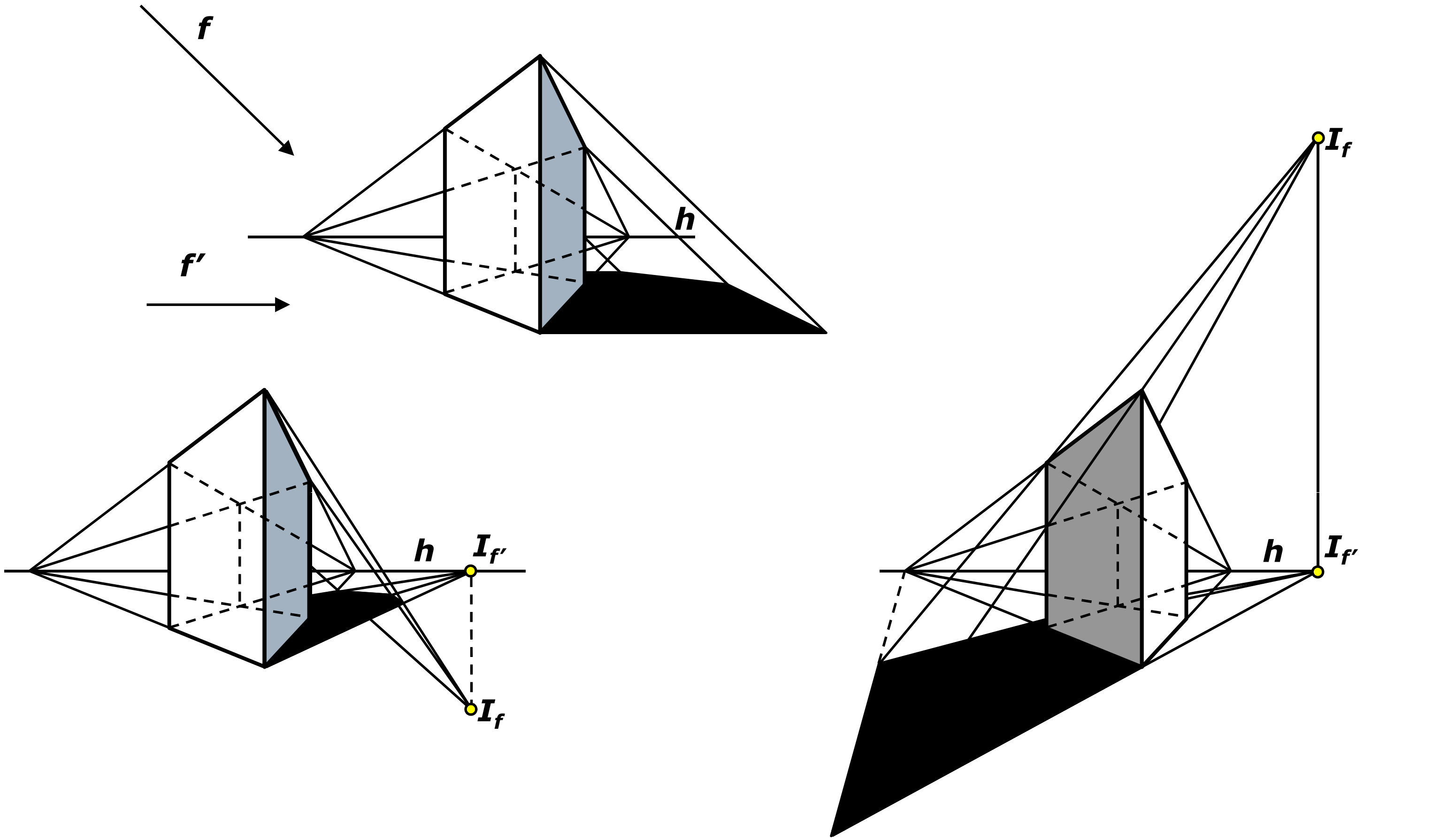


Shadow in Perspective

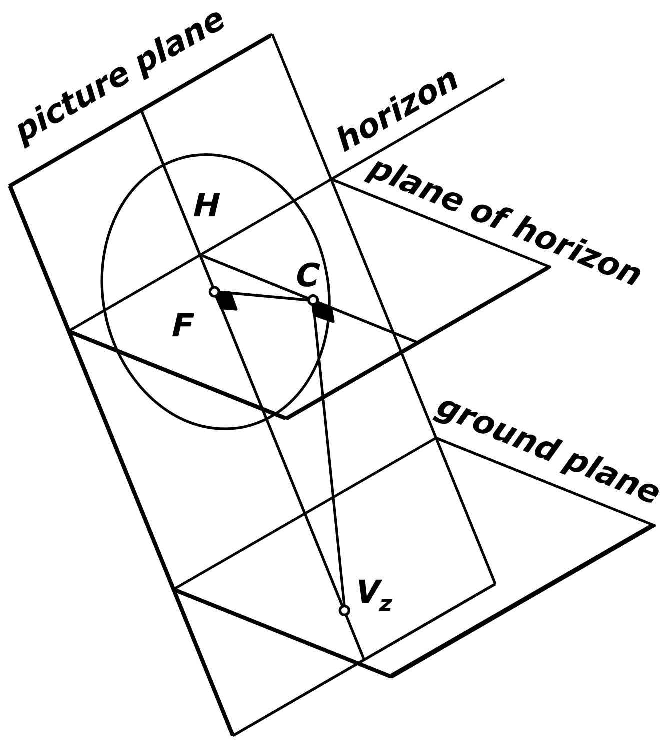


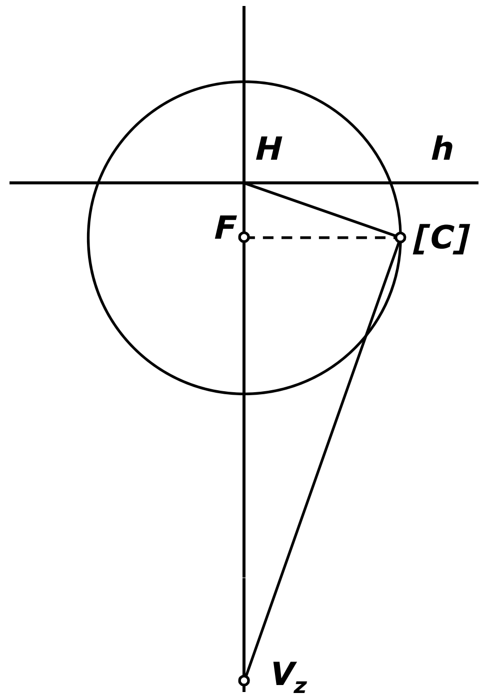
http://www.math.ubc.ca/~cass/courses/m309-03a/m309-projects/endersby/Antisolarpoint.html

Shadow Types in Perspective



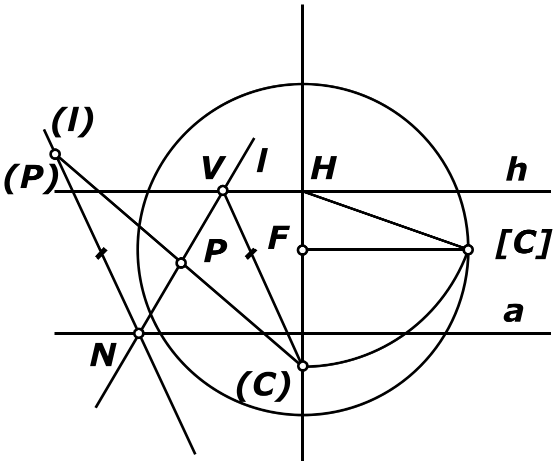
Perspective with Slanting Picture Plane



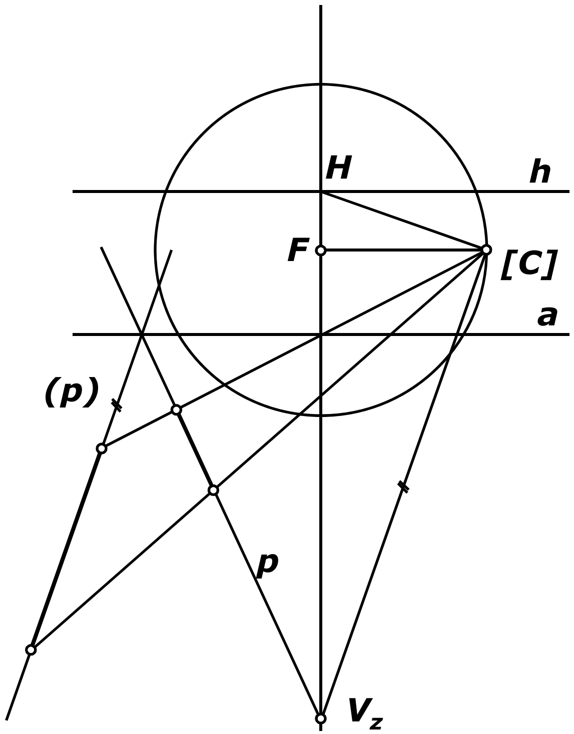


Constructions with Slanting Picture Plane

Rotation of the ground plane



Rotation of a vertical line



Perspective with Slanting Picture Plane