BUTE Department of Architectural Representation

Geometrical Constructions 1

by Pál Ledneczki Ph.D.

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„Let no one destitute of geometry enter my doors”

ἀγεωμὲρητος
μηδεὶς
εἰσίτω

RAFFAELLO: *School of Athens*, Plato & Aristotle

Preface

The central purpose of the subject *Geometrical Constructions* is to provide the prerequisites of *Descriptive Geometry* for our students.

At the Budapest University of Technology and Economics, Faculty of Architecture we teach applied geometry; traditionally called *Descriptive Geometry* for the B.Sc. students in the first and second semester. At *Descriptive Geometry* students are supposed to be familiar with 2D geometrical constructions, with mutual positions of spatial elements in 3D and suitably skilful at the use of drawing instruments.

The subject material of *Geometrical Constructions* is organized as follows.

The ***Lecture Notes*** contains the outline of the course. This booklet is not for sparing note-taking, students are supposed to take their own notes on the lectures.

The ***Student Activity Manual*** is a collection of worksheets. Since the worksheets will be used both on the lectures and practical sessions, this manual should be ready at hand on the practical sessions and advisable at the lectures.

The three items; your handwritten lecture notes, the printed Lecture Notes and the collection of worksheets compose your personal ***Folder*** on *Geometrical Constructions*.

Drawing Instruments

For note-taking and constructions

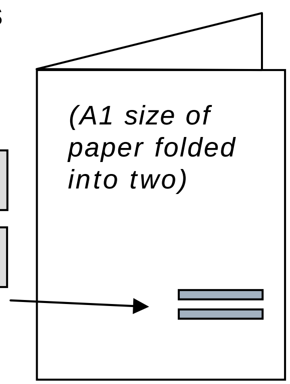
- loose-leaf white papers of the size A4
- pencils of 0.3 (H) and 0.5 (2B)
- eraser, white, soft
- set square (two triangles, 20 cm)
- pair of compasses with joint for pen
- set of color pencils
- drawing board

For drawings (assignments)

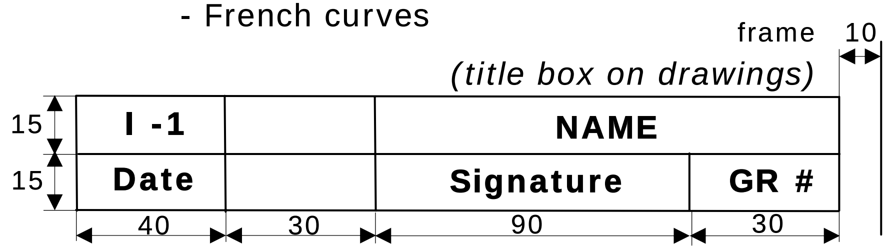
- three A2 sheets of drawing paper (594 mm × 420 mm)
- set of drawing pens (0.2, 0.4, 0.7)
- set square (two triangles, 30 cm)
- A1 sheet of drawing paper for drawings
- French curves

Name

Architectural Representation



(title box on drawings)

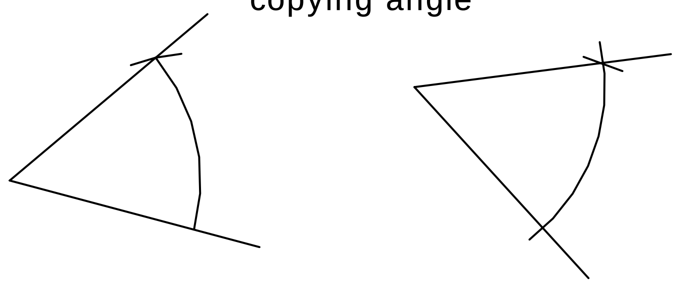


lettering

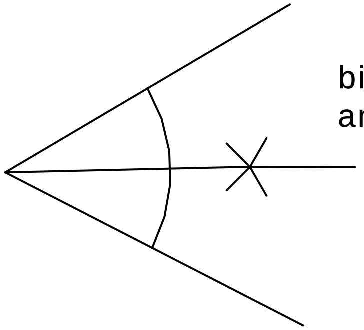
abcdefghijklmnopqrstuvwxyz1234567890
ABCDEFGHIJKLMNOPQRSTUVWXYZ

Use of Set Squares and Pair of Compasses

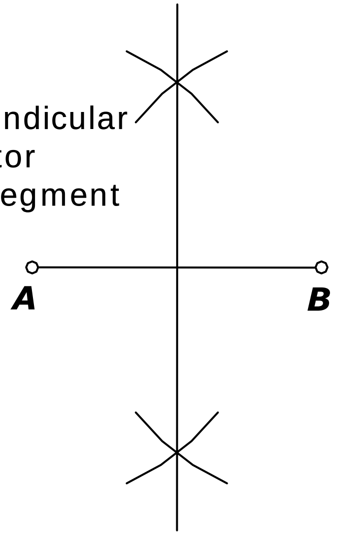
copying angle

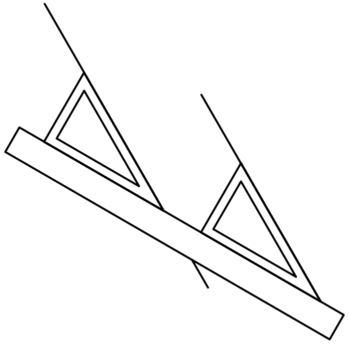


bisecting
an angle

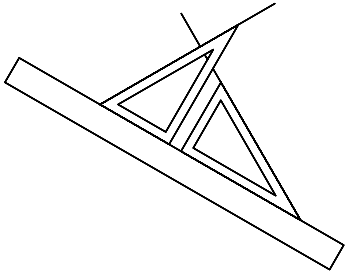


perpendicular
bisector
of a segment



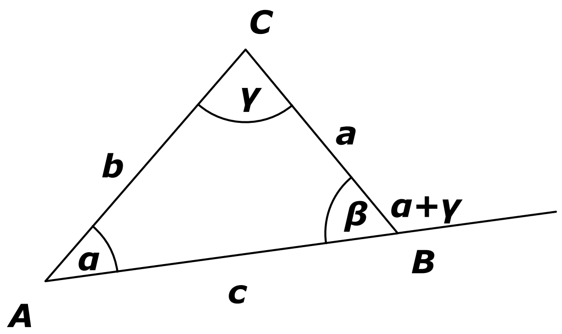


parallel lines by
sliding ruler



perpendicular
line by rotating ruler

Triangles, Some Properties



Triangle inequalities

$$a+b > c, \quad b+c > a, \quad c+a > b$$

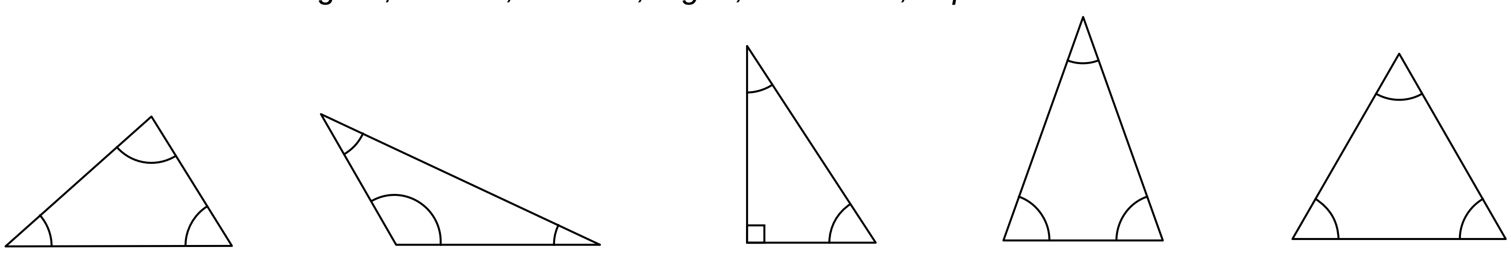
Sum of interior angles

$$\alpha + \beta + \gamma = 180°$$

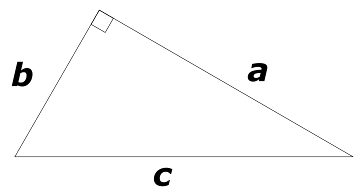
Any exterior angle of a triangle is equal to the sum of the two interior angles non adjacent to the given exterior angle.

About the sides and opposite angles, if and only if $b > a$ than $\beta > \alpha$ for any pair of sides and angles opposite the sides.

Classification of triangles; *acute, obtuse, right, isosceles, equilateral*



Theorems on Right Triangle



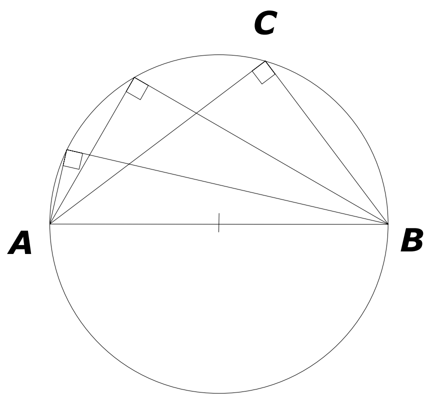
$$a^2 + b^2 = c^2$$

Theorem of Pythagoras

In a right-angled triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.

Converse:

If the sum of the squares of two sides in a triangle is equal to the square of the third side, then the triangle is a right triangle.



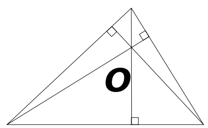
Theorem of Thales

In a circle, if we connect two endpoints of a diameter with any point of the circle, we get a right angle.

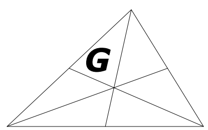
Converse:

If a segment ***AB*** subtends a right angle at the point ***C***, then ***C*** is a point on the circle with diameter ***AB***.

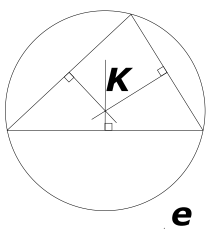
Triangles, Special Points, Lines and Circles



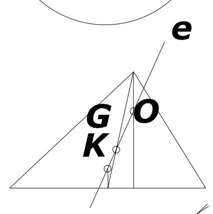
The three altitudes of a triangle meet at a point ***O***. This point is the *orthocenter* of the triangle. The orthocenter of a right triangle is the vertex at the right angle.



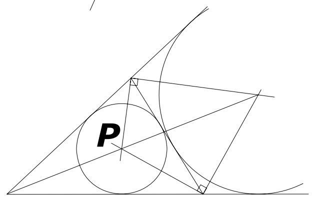
The three medians of a triangle meet at a point ***G*** (point of gravity). This point is the *centroid* of the triangle. The centroid trisect the median such that the segment connecting the vertex and the centroid is the 2/3-rd of the corresponding median.



The three perpendicular bisectors of the sides of a triangle meet at a point ***K***. This point is the *center of the circumscribed circle* of the triangle.

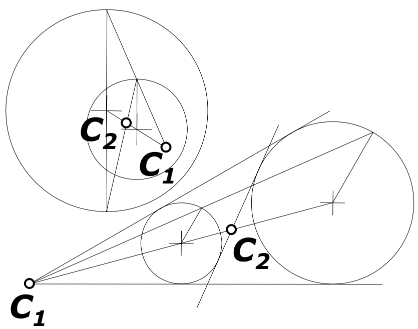


The three points ***O***, ***G***, and ***K*** are collinear. The line passing through them is the *Euler's line*. The ratio of the segments ***OG*** and ***GK*** is equal to 2 to 1.



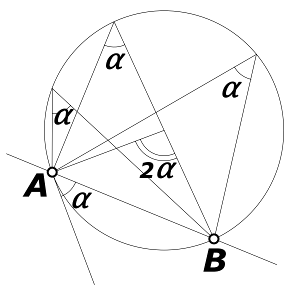
The three bisectors of angles of a triangle meet at a point ***P***. This point is the *center of the inscribed circle*. (Two exterior bisectors of angles and the interior bisector of the third angle also meet at a point. About this point a circle tangent to the lines of the triangle can be drawn.)

Theorems on Circles



Circles are similar

Two circles are always similar. Except the case of congruent or concentric circles, two circles have two centers of similitude C_1 and C_2.



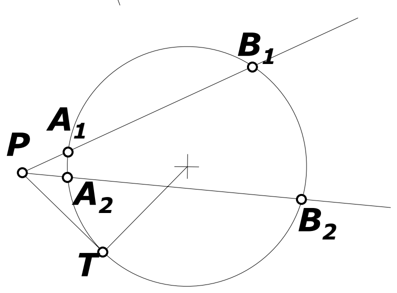
Theorem on the angles at circumference and at center

The angle at the center is the double the angle at the circumference on the same arc.

Corollaries:

A chord subtends equal angles at the points on the same of the two arcs determined by the chord.

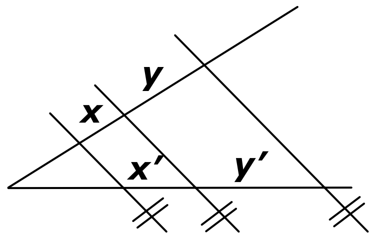
In a cyclic quadrilateral, the sum of the opposite angles is 180°.



Circle power

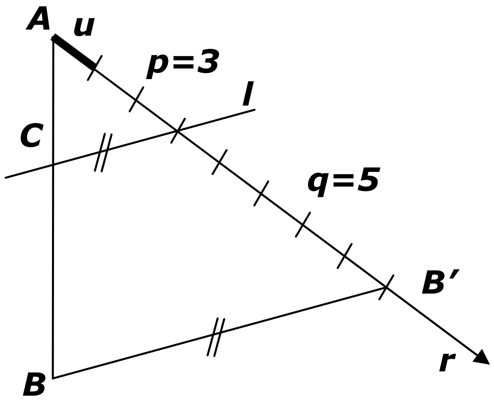
On secants of a circle passing through a point ***P***, the product of segments is equal to the square of the tangential segment: $PA_1 * PB_1 = PA_2 * PB_2 = PT^2$

Parallel Transversals, Division of a Segment



Theorem on parallel transversals: the ratios of the corresponding segments of arms of an angle cut by parallel lines are equal.

$$x:y = x':y', \qquad x:x' = y:y'$$



Divide the segment ***AB*** *by a point* ***C*** *such that* ***AC:CB = p:q***.
Find the point ***C.***

Solution:

1) draw an arbitrary ray ***r*** from ***A***

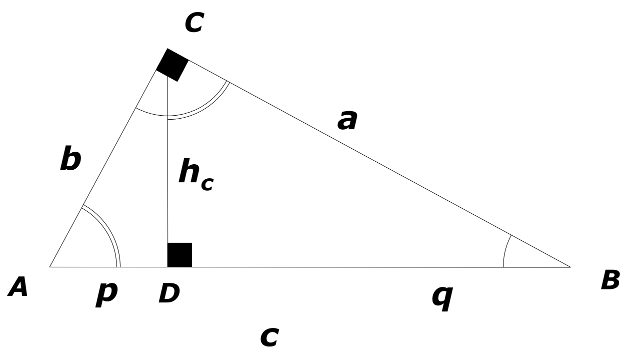
2) measure an arbitrary unit ***u*** from ***A*** onto the ray ***p + q*** times, get the point ***B'***

3) connect ***B*** and ***B'***

4) draw parallel ***l*** to ***BB'*** through the endpoint of segment ***p***

5) ***C*** is the point of intersection of ***l*** and ***AB***

Ratios of Segments in Triangle

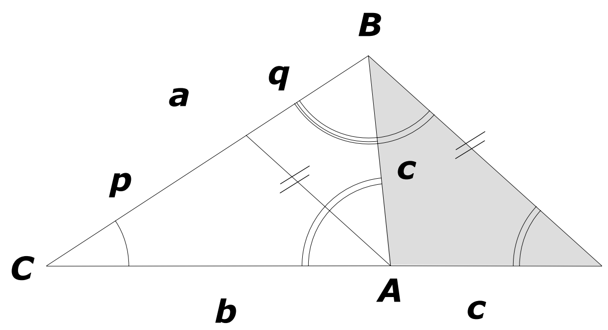


In a right triangle,

$$h_c^2 = pq,\ b^2 = cp,\ a^2 = cq.$$

The altitude assigned to the hypotenuse is the geometrical mean of the two segments of the hypotenuse.

A leg is the geometrical mean of the hypotenuse and the orthogonal projection of the leg on the hypotenuse.



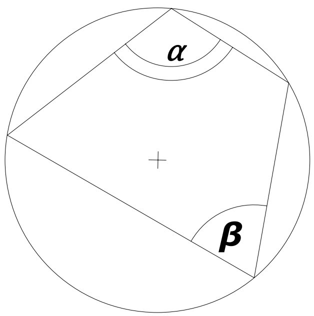
In an arbitrary triangle,

$$\frac{p}{q} = \frac{b}{c}.$$

The bisector of an angle in a triangle divides the opposite side at the ratio of the adjacent sides.

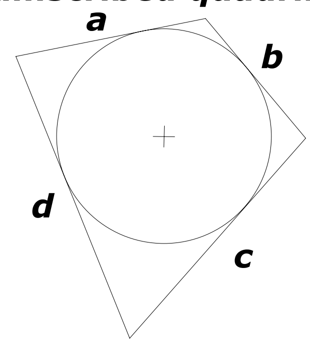
Quadrilaterals

Cyclic quadrilateral



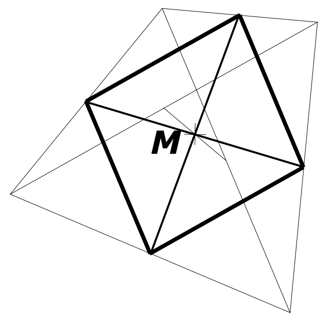
$\alpha + \beta = 180°$

Circumscribed quadrilateral



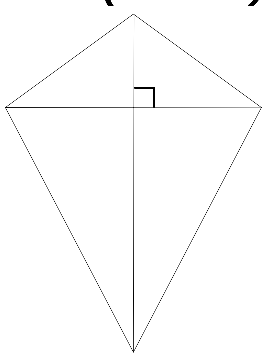
$a + c = b + d$

Diagonals and bimedians

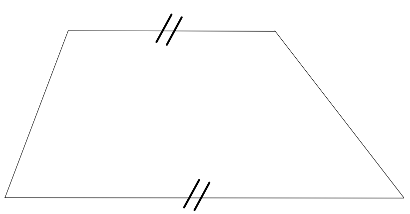


coincidence of midpoints

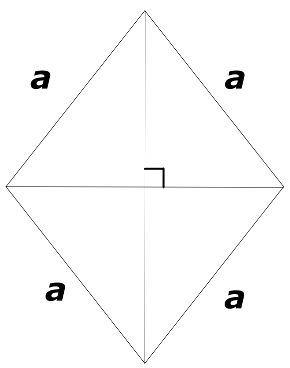
Kite (Deltoid)



Trapezium



Rhombus



Euclid of Alexandria (about 325BC-265BC)

Euclidean construction (instruments: *straight edge and a pair of compasses*):

We may fit a ruler to two given points and draw a straight line passing through them

We may measure the distance of two points by compass and draw a circle about a given point

We may determine the point of intersection of two straight lines

We may determine the points of intersection of a straight line and a circle

We may determine the points of intersection of two circles

More than one thousand editions of ***The Elements*** have been published since it was first printed in 1482.

BL van der Waerden assesses the importance of the Elements:

‘Almost from the time of its writing and lasting almost to the present, the Elements has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the "Elements" may be the most translated, published, and studied of all the books produced in the Western world.’

This is a detail from the fresco *The School of Athens* by *Raphael*

http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Euclid.html

How to Solve It?

(according to George Polya, 1887 - 1985)

First. *You have to* ***understand the problem.***

What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
Draw a figure. Introduce suitable notation.
Separate the various parts of the condition. Can you write them down?

Second. *Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a* ***plan of the solution.***

Have you seen it before? Or have you seen the same problem in a slightly different form?
Do you know a related problem? Do you know a theorem that could be useful?
Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.
Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
Could you restate the problem? Could you restate it still differently? Go back to definitions.

http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Polya.html

How to Solve It?

If you cannot solve the proposed problem do not let this failure afflict you too much but try to find consolation with some easier success, try to solve first some related problems; then you may find courage to attack your original problem again. Do not forget that human superiority consists in going around an obstacle that cannot be overcome directly, in devising some suitable auxiliary problem when the origin alone appears insoluble. Try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Third. ***Carry out your plan.***
Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

Looking Back

Fourth. ***Examine the solution obtained.***
Can you check the result? Can you check the argument?
Can you derive the solution differently? Can you see it at a glance?
Can you use the result, or the method, for some other problem?

Geometrical Constructions

Summary of problem solving method:

Sketch

Draw a sketch diagram as if the problem was solved

Plan

Try to find relations between the given data and the unknown elements, make a plan

Algorithm

Write down the algorithm of the solution

Discussion

Analyze the problem in point of view of conditions of solvability and the number of solutions

Chapter Review

Vocabulary

Acute angle
Right angle
Obtuse angle
Straight angle
Reflex angle
Complementary angle
Supplementary angle
Protractor
Vertical pair of angles
Parallel
Perpendicular
Bisector
Line
Collinear
Coplanar
Concurrent
Equilateral triangle
Isosceles Triangle
Right triangle
Base of triangle

Leg of right triangle
Hypotenuse
Interior angle
Exterior angle
Vertex, vertices
Sides
Altitudes of a triangle
Orthocenter
Medians of a triangle
Centroid of a triangle
Point of gravity
Circumscribed circle of a triangle
Inscribed circle of a triangle
Radius of a circle
Diameter of a circle
Chord of a circle
Secant of a circle
Angle at center in a circle
Angle at circumference
Arithmetical mean
Geometrical mean

Quadrilateral
Square
Rectangle
Rhombus
Parallelogram
Trapezoid
Isosceles trapezoid
Kite
Cyclic quadrilateral
Circumscribed quadrilateral
Diagonal
Bimedian
Straight edge
Pair of compasses
Sketch
Accuracy
Adjacent
Convex
Concave
Orthogonal projection

Chapter Review

Ideas, Theorems

Triangle inequalities
Sum of internal angles in a triangle
Theorem of Pythagoras and its converse
Theorem of Thales and its converse
The three medians of a triangle meet at a point
The three altitudes of a triangle meet at a point
The three bisectors of sides of a triangle meet at a point
The bisectors of angles of a triangle meet at a point
The centroid, orthocenter and center of circumscribed circle of a triangle are collinear
Euler's line of a triangle
Circle power
Parallel transversals
Proportional division
The bisector of an angle in a triangle divides the opposite side
Diagonals and bimedians in a quadrilateral
Euclidean construction

Exercises, Basic Constructions

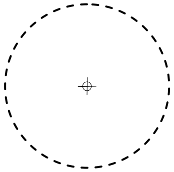
1. Bisect an angle.
2. Copy an angle.
3. The sum and the difference of two angles are given. Find the two angles.
4. Show that the bisector of an angle and the bisector of the supplementary angle are perpendicular.
5. Bisect a segment.
6. Double a given segment by means of compass.
7. Construct a triangle with the ratio of angles 1:2:3.
8. What kind of triangle has the side lengths of 3,3 and 7?
9. Construct the triangle with the sides 5, 12, and 13.
10. The distances of a point located between the two arms of a right angle are a and b. Find the distance (formula) of the point and the vertex of the right angle.
11. One of the angles of a right triangle is 30° and one of the sides is 6. Find all solutions.
12. The hypotenuse of an isosceles right triangle is 4. Find (construct) the legs.
13. The common chord of a pair of intersecting circle is 6. Find (construct) the distance of the centers, if the radii are 4 and 5.
14. The radius of a circle is 25. A pair of parallel chords are 14 and 40. Find the distance of chords. Draw sketch and calculate.
15. Find the center of a circle by using a right triangle ruler. Write dawn the algorithm.
16. Construct the “2” shape inscribed in a rectangle.
17. Let a segment and a point of the segment be given. Draw a line through the point such that the orthogonal projection of the segment on the straight line is equal to a given length (shorter than the given segment).

Exercises, Basic Constructions

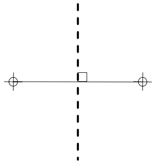
18. Let three points be given, one vertex, the orthocenter and the centroid of a triangle. Find the triangle.
19. Construct triangle determined by one of the sides, the angle opposite the given side and the height assigned to the given side
20. Let a pair of intersecting circle be given. Through one of the points of intersection draw a line segment in the circles and connect the endpoints with the other point of intersection. The angle formed by them is constant.
21. Construct triangle, the height, median and angle assigned to a side are given.
22. Find the circle, passing through two points and tangent to a line. (Circle power)
23. Divide a segment at the ratio of 2:3:5. (Parallel transversals)
24. Construct the points O, G, C, P.
25. In a right triangle, if AD = 24 and BD = 9, find CD; if AD = 6 and CD = 4, find BD; if AB = 20 and CD = 6, find BD.
26. The diameter of a circle AB is 36 cm. The points E and F trisect the diameter and they are the pedal points of the vertices C and D of rectangle, whose diagonal is the diameter AB. Find the length of sides of rectangle. (Calculate and construct with the scale 1:6. this is the most load bearable intersection of lumber.)
27. Cut a convex quadrilateral along the bimedians into four parts. Show that the parts can arrange into a parallelogram.
28. Construct parallelogram if two sides and one of the diagonals are given.
29. Construct parallelogram if one side and two diagonals are given.
30. The area of the quadrilateral determined by the midpoints of the sides of a convex quadrilateral is equal to 180 cm^2. Find the area of the original quadrilateral.

Determining Locus

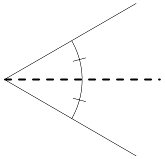
Locus in Latin means *location*. The plural is *loci*. A *locus of points* is the set of points, and only those points, that satisfy the given conditions.



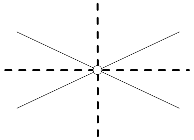
The locus of points at a given distance from a given point is a circle whose center is the given point and whose radius is the given distance.



The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining the two points.



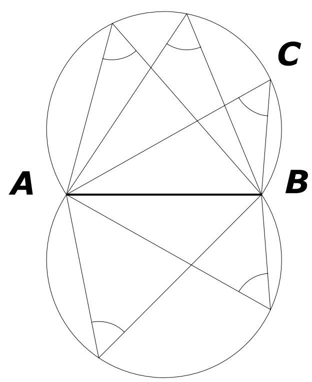
The locus of points equidistant from the sides of a given angle is the bisector of the angle.



The locus of points equidistant from two given intersecting lines is the bisectors of the angles formed by the lines.

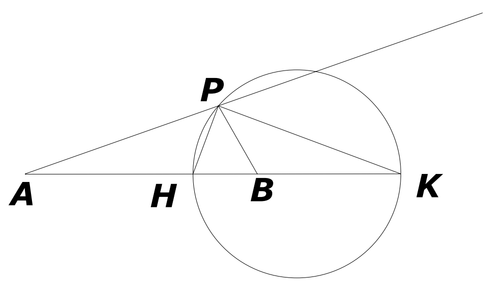
Etc.

Theorems on Loci



If a segment ***AB*** subtends a given angle at the point ***C***, then the locus of ***C*** consists of two arcs of circles of the same radius, symmetrical with respect to the segment.

This theorem is the converse of the theorem on angles at circumference, and generalization of the converse of Thales theorem.



A and ***B*** are fixed points. ***P*** is a moving point such that λ **= *PA : PB*** is constant, than the locus of ***P*** is a circle.

Hint to the proof: ***PH*** and ***PK*** are angle bisectors

$\angle$***HPK*** = 90°

An angle bisector in a triangle divides the opposite side at the ratio of the adjacent sides

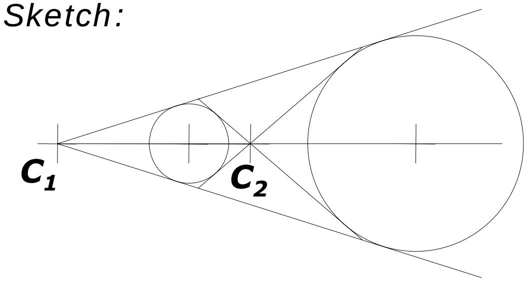
$$\frac{AH}{HB} = \frac{AP}{PB} = \lambda$$ constant, ***H*** and ***K*** are fixed.

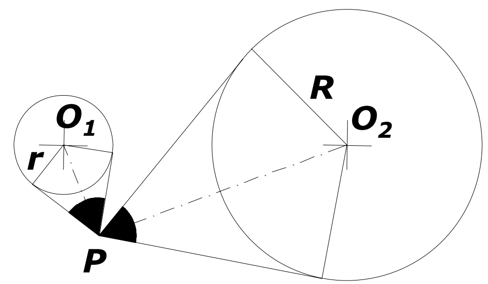
(This circle is called *Apollonian* circle.)

Exercise on Locus

Find the locus of points at which two given circles subtend equal angles.

Sketch:





The centers of homothecy are obviously points of the locus.

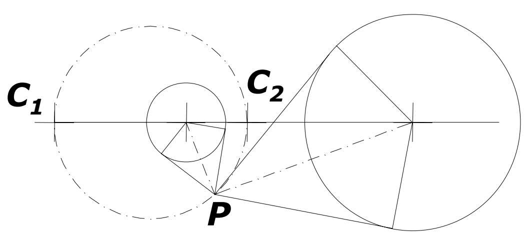
Because of the similarity of circles,

$$\frac{PO_1}{PO_2} = \frac{r}{R} = \text{ constant.}$$

According to the theorem of Apollonius, the locus is a circle.

Algorithm:
1. find the centers of similarity
2. draw the circle with the diameter C_1C_2

Construction:



Discussion:
1) circles with equal radii; perpendicular bisector of O_1O_2
2) circles with different radii, disjoint or touching; see the construction
3) circles with different radii, partially overlapping; the arc of circle outside the union of the given circles
4) circles with different radii, one contains the other, tangents; the point of contact
5) circles with different radii, no point in common, overlapping; no solution

Exercises, Loci Problems

1. Represent the points, whose distance from a point is less than a given length.
2. Represent the points, whose distance from a segment is less than a given length.
3. Represent the points, whose distance from a pair of intersecting lines is equal to a given length.
4. Represent the points, equidistant from a pair of parallel lines.
5. Represent the points, equidistant from a pair of intersecting lines.
6. Construct the points, equidistant from three non-concurrent lines.
7. What is the locus of the vertices C of a triangle ABC, whose A and B vertices are fixed and the and the radius of the circumscribed circle of the triangle ABC is a given length?
8. Construct min. 8 points, whose sum of distances from a pair of points (focus, plural: foci) is a given length (grater than the distance of the given points).
9. Construct min. 8 points, whose difference of distances from a pair of points is a given length (shorter than the distance of the given points).
10. Construct min. 7 points equidistant from a point and a line (not passing through the points).
11. What is the locus of points, for which the ratio of distances prom a pair of points is 1:2. (Apollonius)
12. Let the distance of a point and a line be 4 cm. Construct min. 7 points of the locus of points whose distance from the line towards the point is equal to 3 cm (4 cm, 5 cm, 2 cm). (http://en.wikipedia.org/wiki/Conchoid_of_Nichomedes)
13. Construct the isosceles triangle determined by a leg and the height/median assigned to the leg.
14. The perimeter and the angle lying on the base of an isosceles triangle are given. Find the triangle.
15. A pair of intersecting lines and a point between the lines are given. Find the circles passing through the point and tangent to the lines.
16. Construct right triangle determined by the hypotenuse and the sum/difference of the legs.
17. Find the locus of points, whose sum of distances from the arms of a right-angle is equal to a given length.
18. Construct the inscribed rectangle of e circle whose perimeter is equal to a given length.

Transformational Geometry

By a transformation of the plane, we mean a one-to-one correspondence ***P* ↔*P′*** among all the points in the plane, that is, a rule for associating pairs of points with the understanding that each pair has a first member ***P*** and a second member ***P′*** and that every point occurs as the first member of just one pair and also as the second member of just one pair. Points that are assigned to themselves are called *invariant* or *fixed* points of the transformation.

Transformations we shall discuss in this course:

Isometries

direct (displacement, sense-preserving)
- *rotation*
- *translation*

opposite (reversal, sense-reversing)
- *reflection*

Similarities

homothecy (dilatation)

Affinities

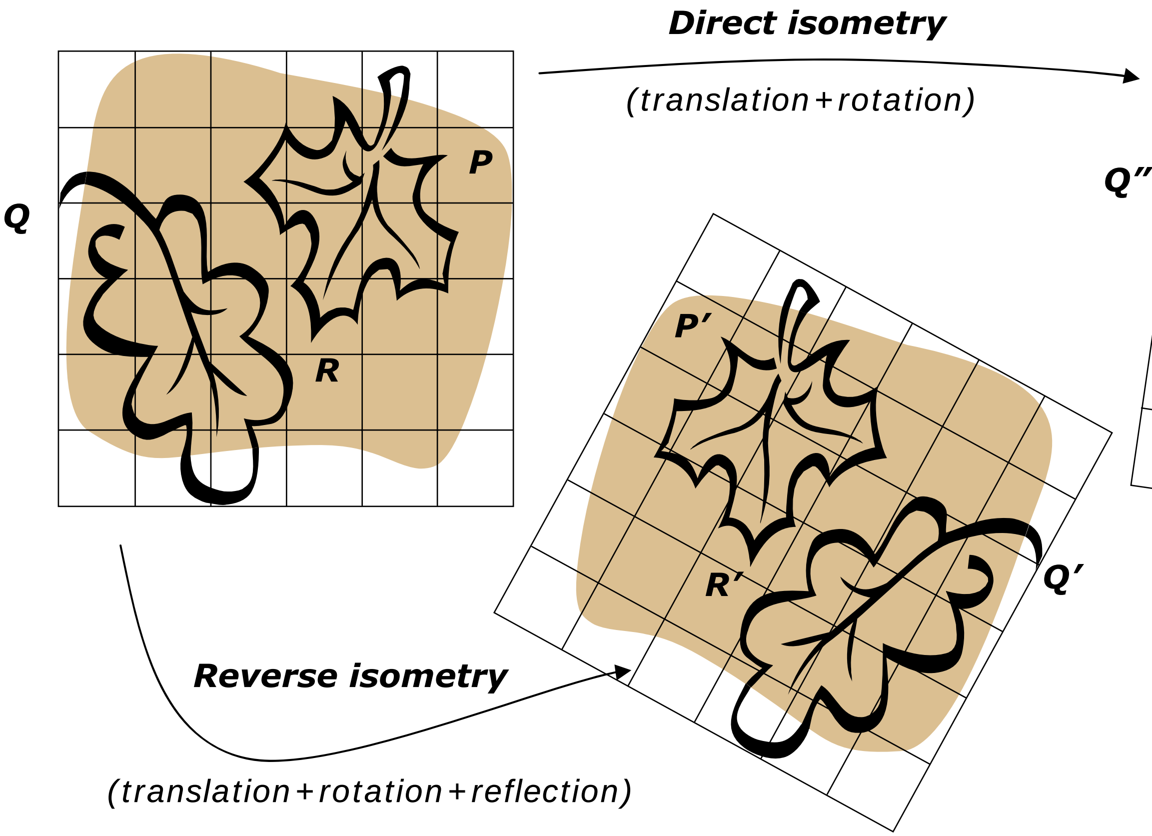
axial

Collineations

central-axial

(Non-linear transformations)

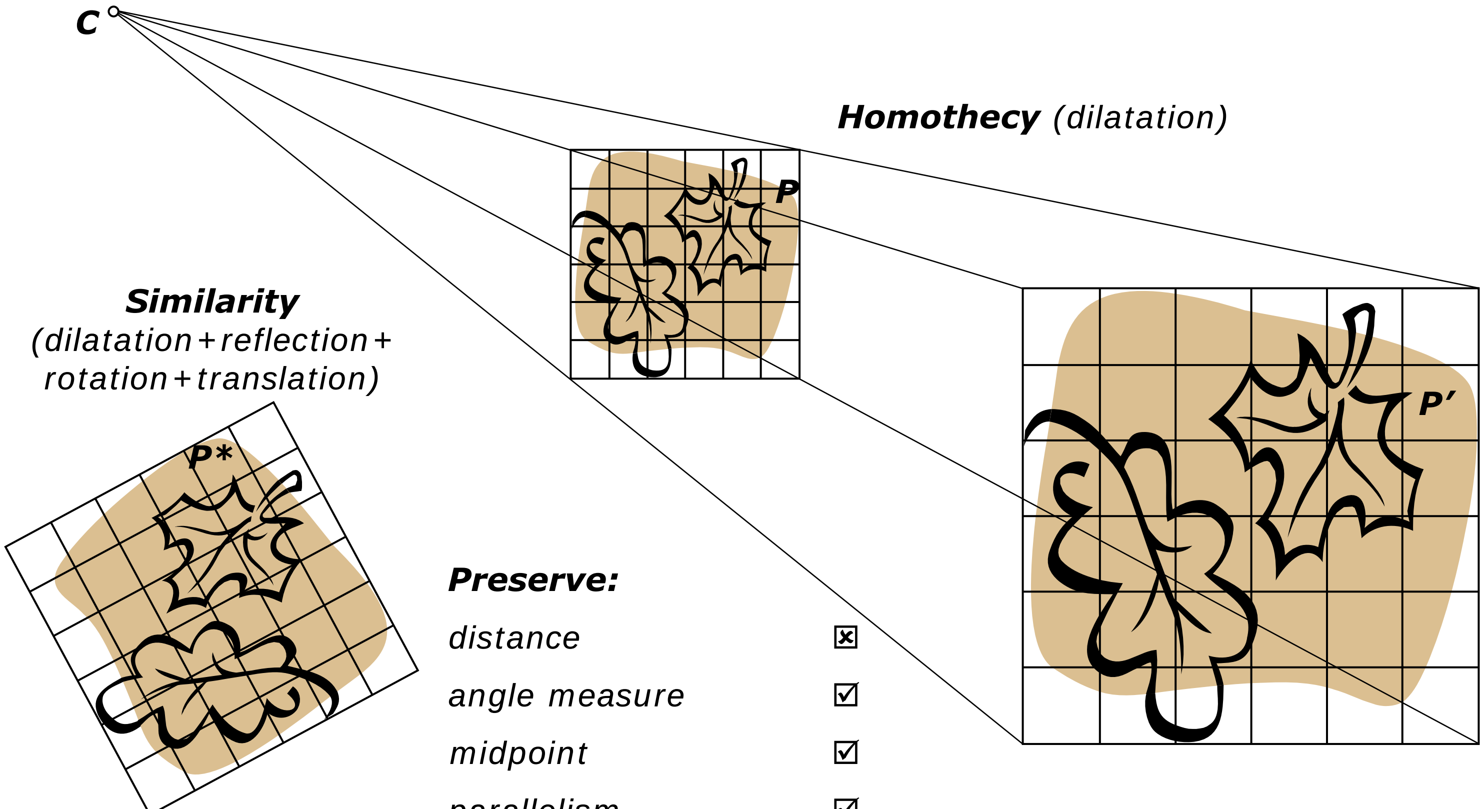
Isometries



Preserve:

distance	☑
angle measure	☑
midpoint	☑
parallelism	☑
collinearity	☑

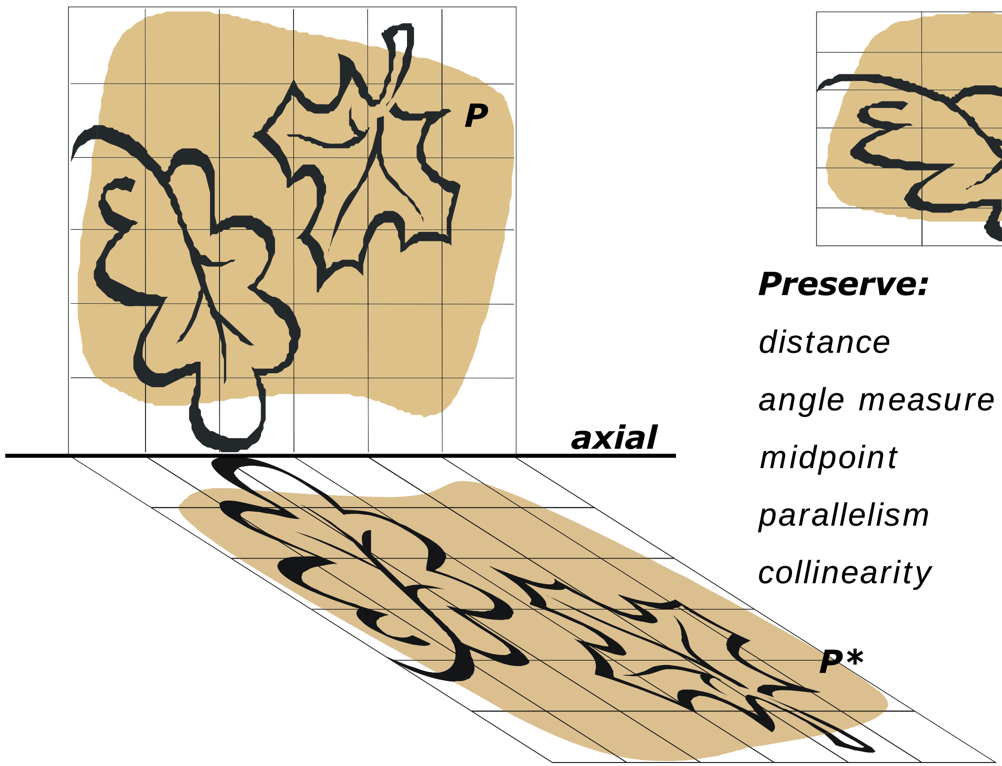
Similarities



Preserve:

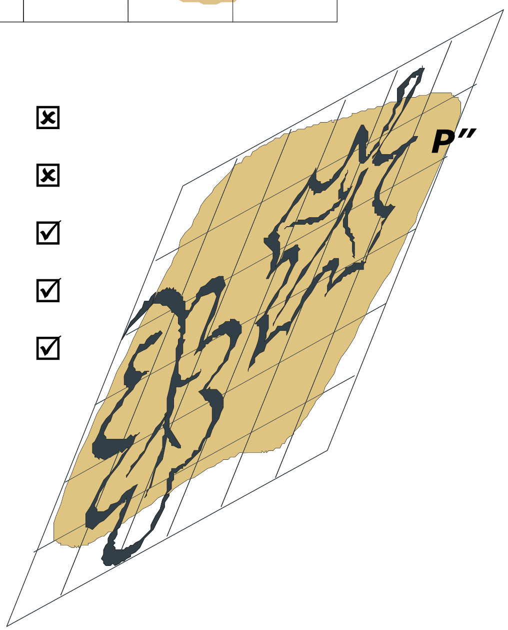
distance	☒
angle measure	☑
midpoint	☑
parallelism	☑
collinearity	☑

Affinities

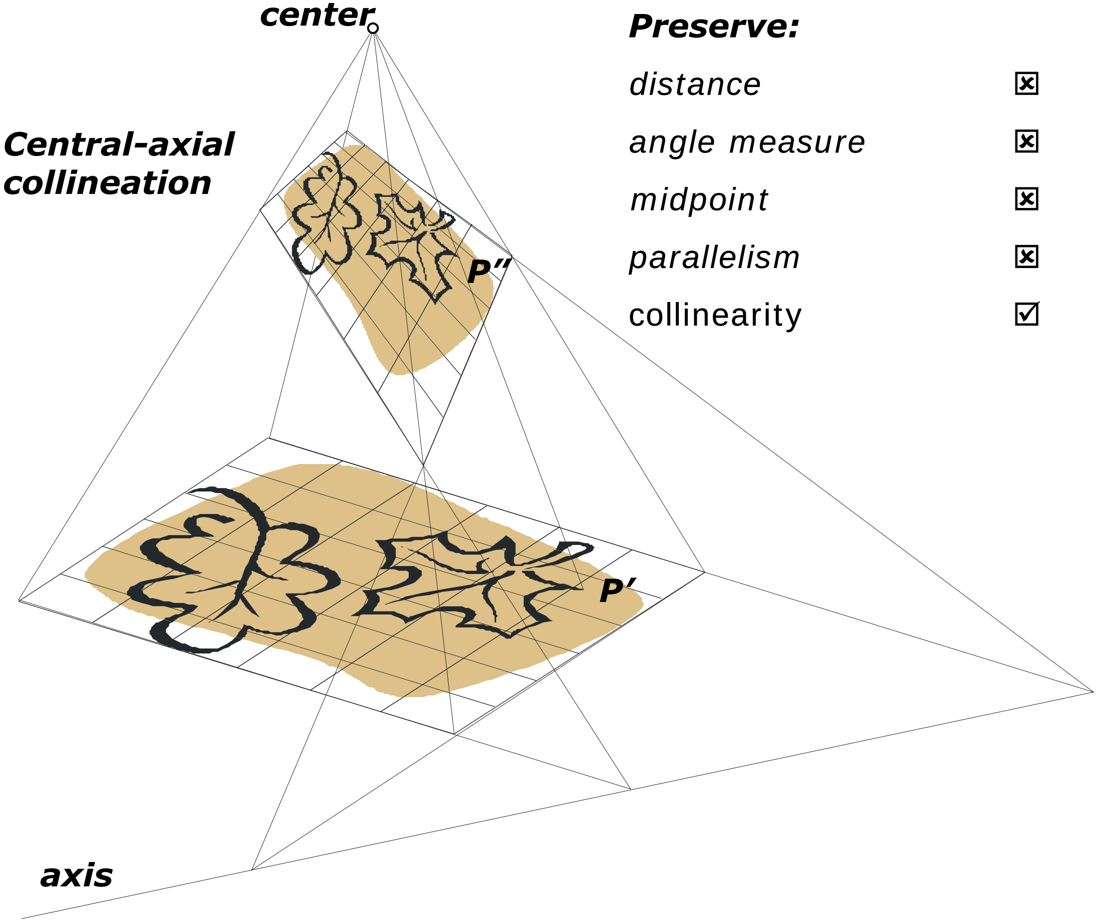


Preserve:

distance	☒
angle measure	☒
midpoint	☑
parallelism	☑
collinearity	☑

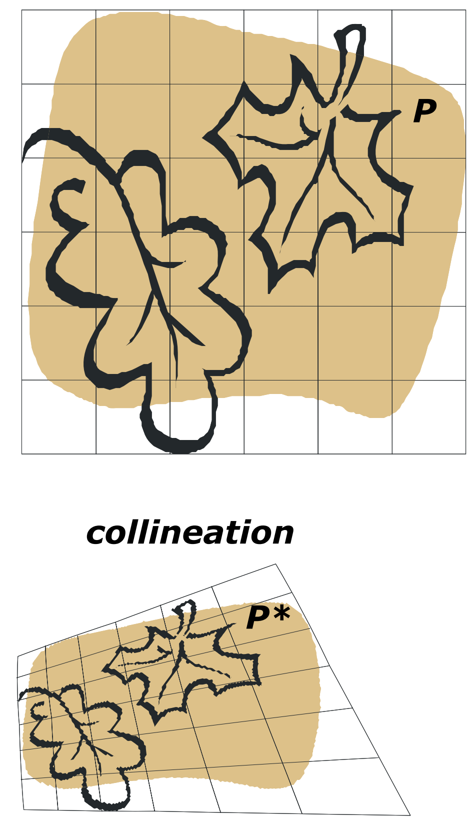


Collineations

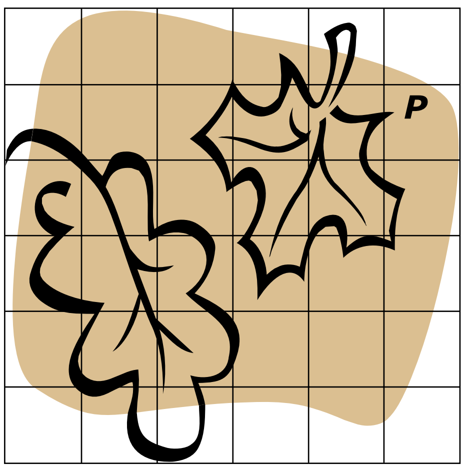


Preserve:

distance	☒
angle measure	☒
midpoint	☒
parallelism	☒
collinearity	☑

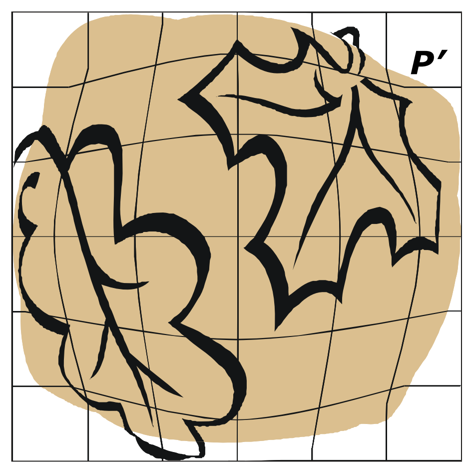


(Non-linear transformation)



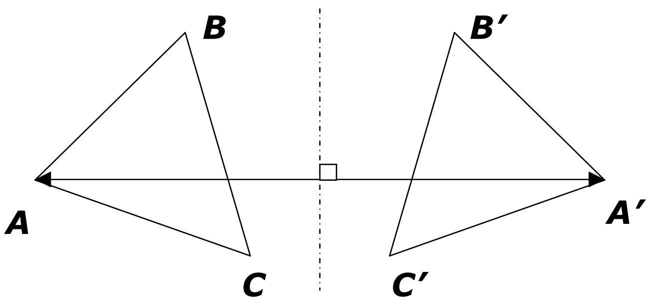
Topological

Any transformation preserving neighborhood is called topological.



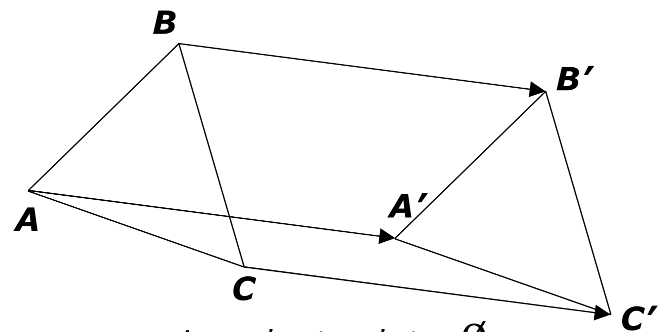
Isometries, Invariant Points

Reflection



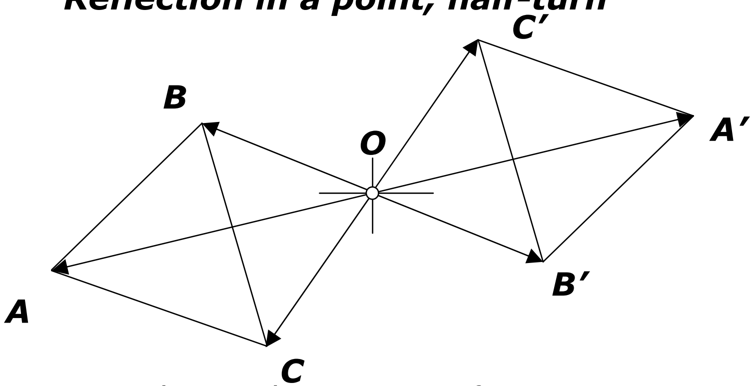
Invariant points: points of the mirror line

Translation



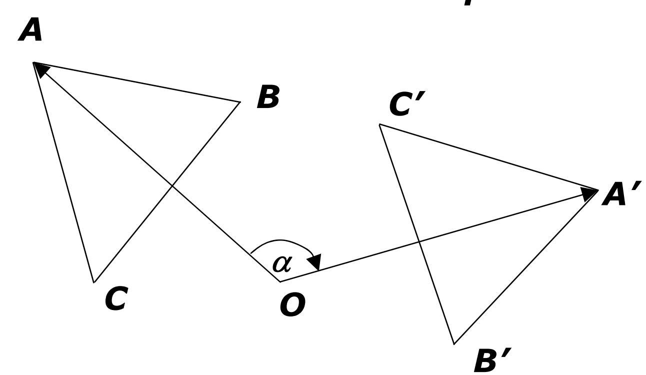
Invariant points: Ø

Reflection in a point, half-turn



Invariant point: center of symmetry

Rotation about a point



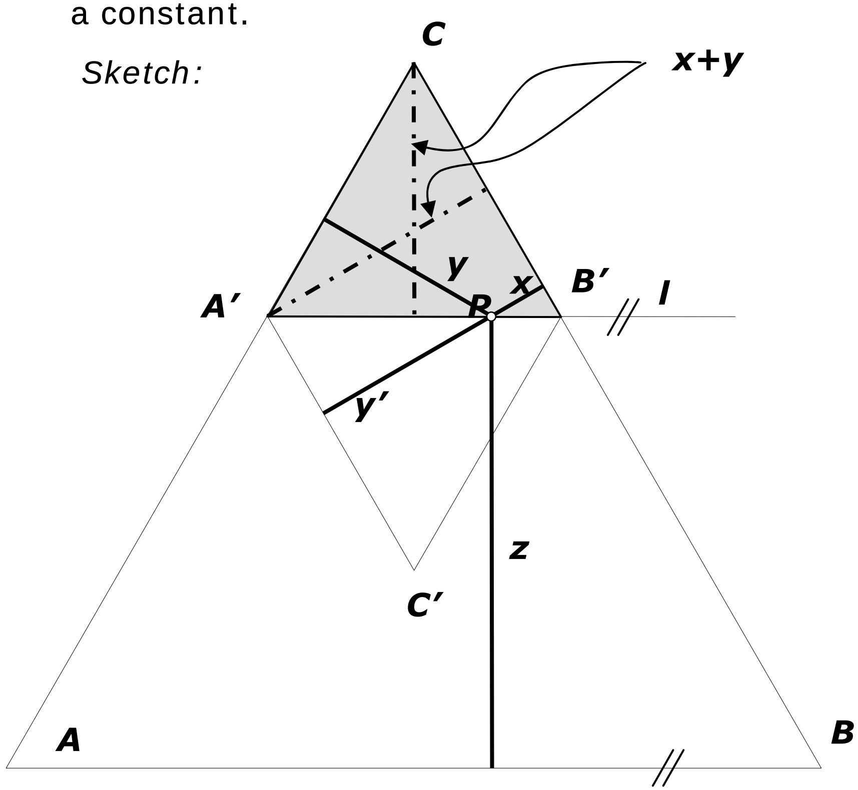
Invariant point: center of rotation

If an isometry has more than one invariant point, it must be either the identity or a reflection.

An Exercise on Isometry

In an equilateral triangle, show that the sum of the distances of an internal point from the sides is a constant.

Sketch:



Proof: 1) draw ***l*** through ***P*** parallel to the base

2) reflect the upper triangle and the segment ***y*** for ***l***

3) ***x+y = x+y′***, ***x*** and ***y′*** are collinear

4) the altitudes in the small triangle through ***A′*** and ***C*** are ***x+y***

5) the sum of the three segments ***x+y+z*** is equal to the length of the altitude of the triangle ***ABC***

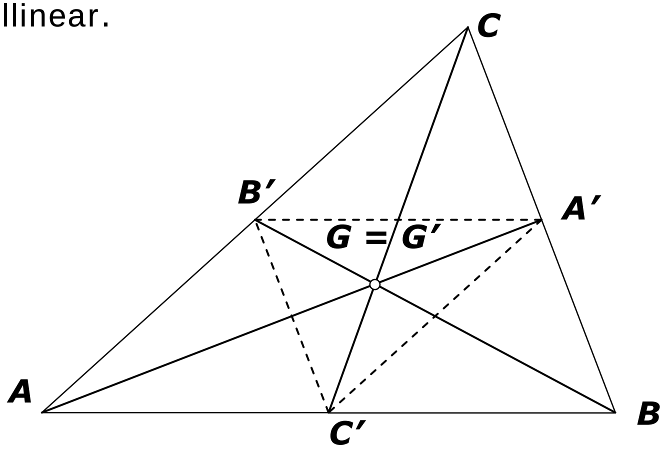
Conclusion: the sum of the distances is the constant length of the altitude of the triangle.

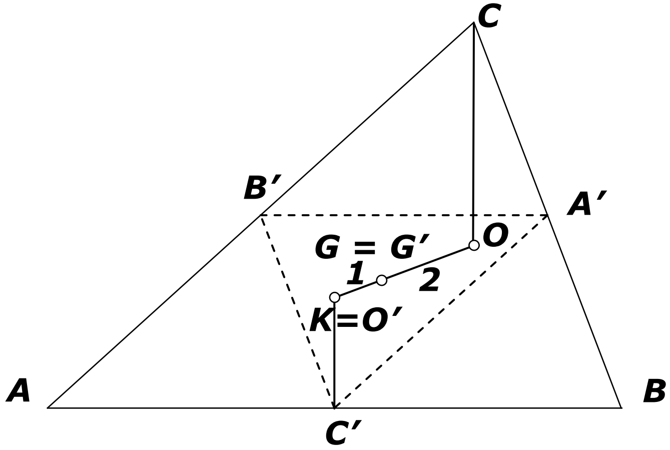
Remark: the statement is true for the points of the sides too.

Problem: In an acute triangle ***ABC***, locate a point ***P*** whose distances from ***A, B, C*** have the smallest possible sum. (Do research on FERMAT point.)

Exercises on Similarity

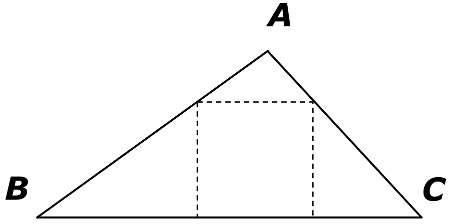
A′,B′,C′ are the mid-points of the sides ***BC, CA, AB*** of a triangle ***ABC***. ***G, O, K*** are the centroid, orthocenter and circumcenter of the triangle ***ABC***. Prove that ***G*** is the center of similitude of the triangles ***A′B′C′*** and ***ABC***, ***K*** is the ortocenter of the triangle ***A′B′C′***, and hence ***G, O*** and ***K*** are collinear.





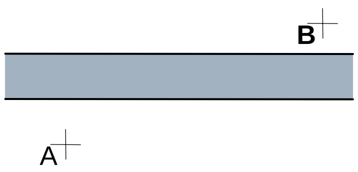
The two triangles ***ABC*** and ***A′B′C′*** are similar with respect to the center ***G***=***G′*** and the ratio of - 2:1, consequently ***O′*** = ***K***, ***G*** and ***O*** are collinear. (The altitudes of the smaller triangle coincide with the perpendicular bisectors of the larger triangle.)

Problem: Given a triangle ***ABC****,* construct a square such that two vertices lie on ***BA*** and ***CA*** respectively, and the opposite side lies on ***BC***.

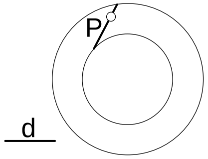


Exercises,Transformation Geometry

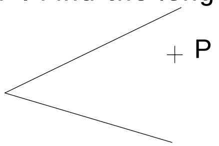
1. Bisect a given segment by means of compass. (Similarity)
2. A straight line, a point on it, and a circle are given. Find the circle tangent to the given line at the given point and also tangent to the given circle. (Dilatation)
3. Let a circle and three directions be given. Find the inscribed triangle such that the sides are parallel to the given directions. (Similarity)
4. Find the circle, passing through two points and tangent to a line. (Reflection for a line)
5. C is a point of the circle whose diameter is AB. D is the reflection of one of the endpoints of the diameter for the point C. Prove that the ABD is an isosceles triangle.
6. The ABC triangle is determined by the vertices A, B, the line of AC and the line of the median passing through A. Construct the triangle.
7. Let a circle and a segment AB be given such that the segment is shorter than the diameter of the circle. Find the chords of the circle parallel to AB.
8. The villages A and B are on different sides of a river. Find the shortest distance between the villages including a bridge, which is perpendicular to the river.



9. A pair of concentric circles, a point between them and a length is given. Find the segment of the given length, connecting the two circles and passing through the point.

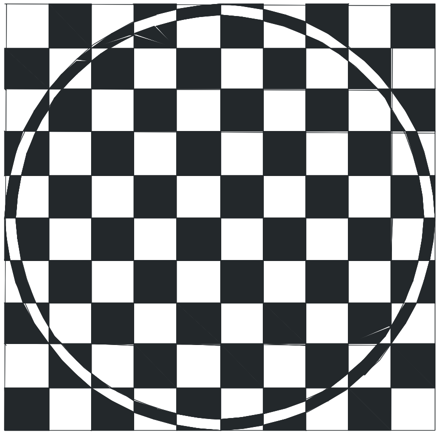


10. The sides of a rectangle are 2.4 dm and 1.8 dm. The area of a similar rectangle is 52cm². Find the length if sides of the rectangle.
11. The marble thrust from P is reflected from the lines one after the other than returns into the position P. Construct the path of the marble.

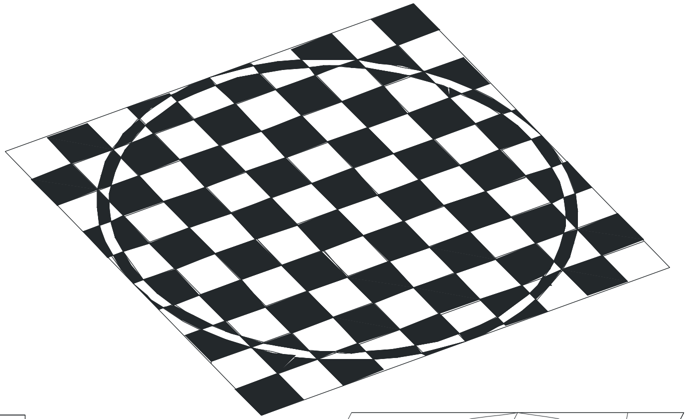


Affine Collineation

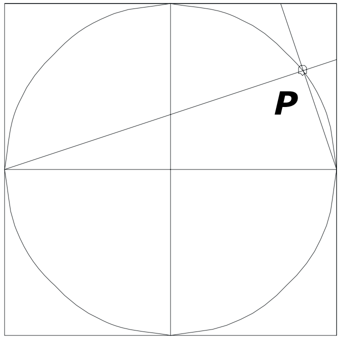
In affine geometry, parallelism plays an important role. An affine transformation preserves neither distances nor angles but the parallelism of lines.

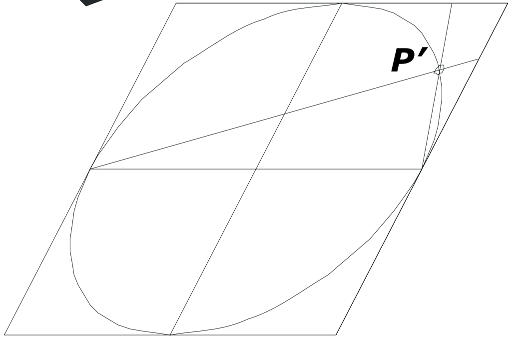


The two chessboards are in affine relation.



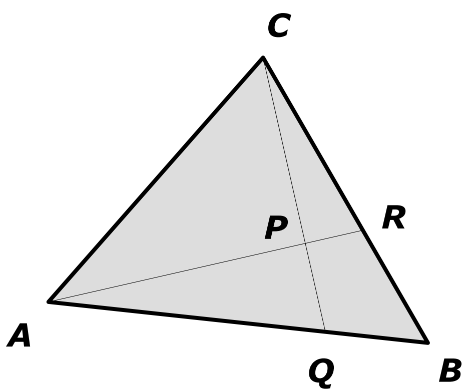
The circle and ellipse are equivalent in affinity.





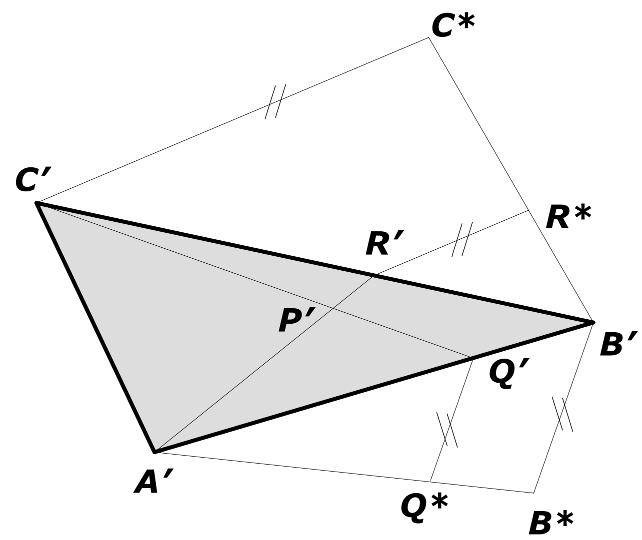
Affine Triangles

Any two triangles are related by a unique affine collineation.



$$\frac{AB}{AQ} = \frac{A'B'}{A'Q'}$$

$$\frac{BC}{BR} = \frac{B'C'}{B'R'}$$



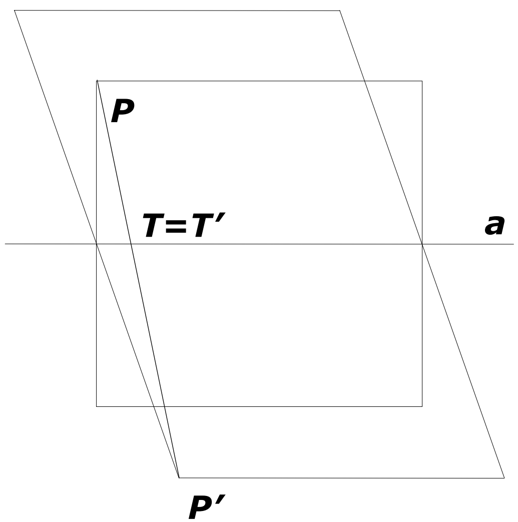
The ratios can be transferred by means of parallel transversals:

$$(AQB) \cong \left(A'Q^{*}B^{*}\right)$$

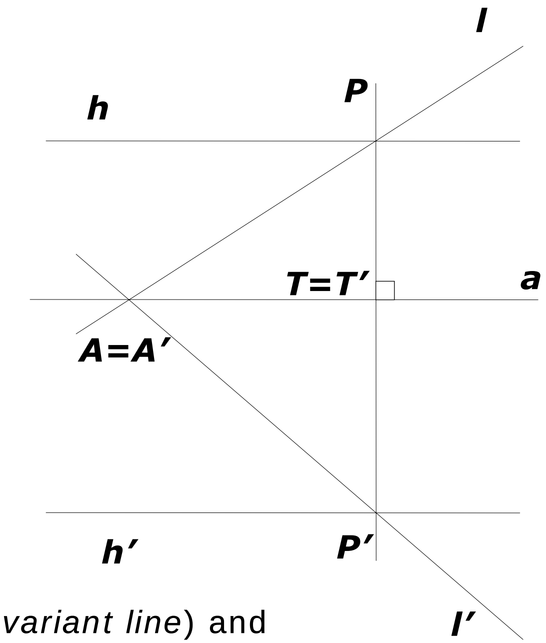
$$(BRC) \cong \left(B'R^{*}C^{*}\right)$$

Axial Affinity

Oblique



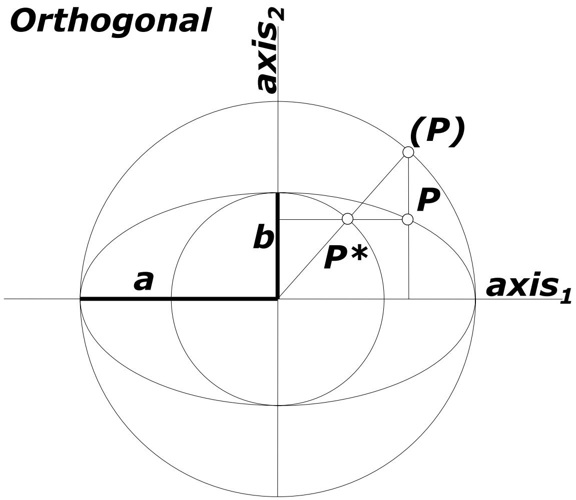
Orthogonal



An axial affinity is uniquely determined by the axis (*invariant line*) and a pair of points. The axis, direction and ratio ***PT:P′T′*** also determines the axial affinity. The axial affinity is called orthogonal if the direction ***PP′*** is perpendicular to the axis.

Circle and Ellipse in Axial Affinity

Orthogonal

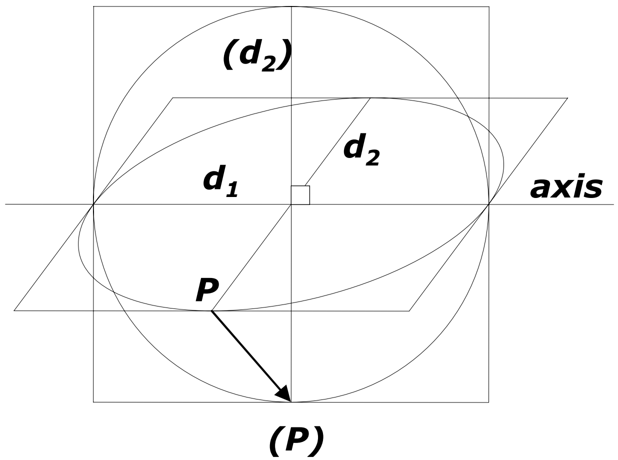


a = *radius of greater circle = half of the major axis*
b = *radius of smaller circle = half of the minor axis*

(P) → ***P*** orthogonal affinity
axis: ***$axis_1$***
ratio: ***b:a***

P* → ***P*** orthogonal affinity
axis: ***$axis_2$***
ratio: ***a:b***

Oblique

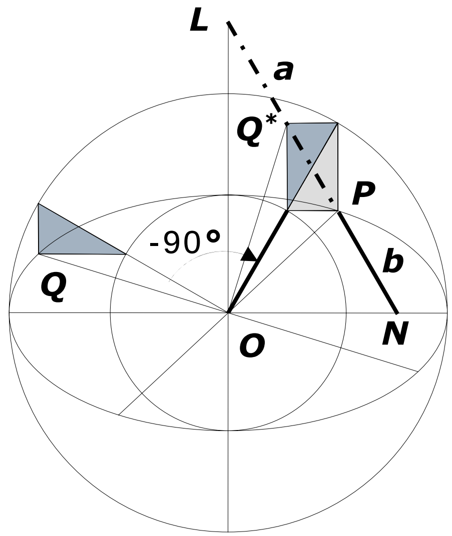


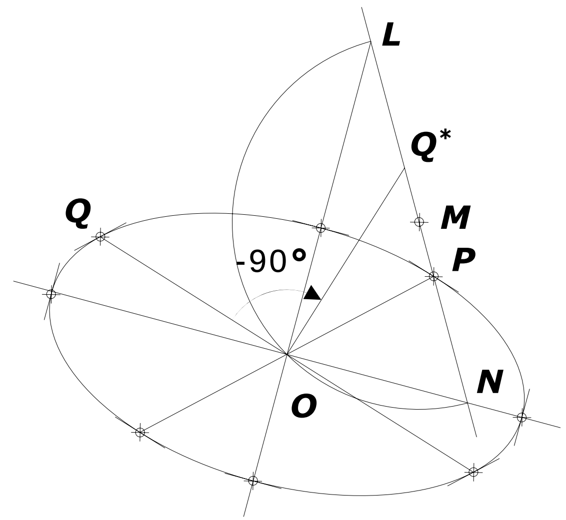
{d_1,(d_2)} pair of perpendicular diameters of the circle →

{d_1,d_2} pair of conjugate diameters of the ellipse

Constructions on Ellipse: Ritz' Costruction

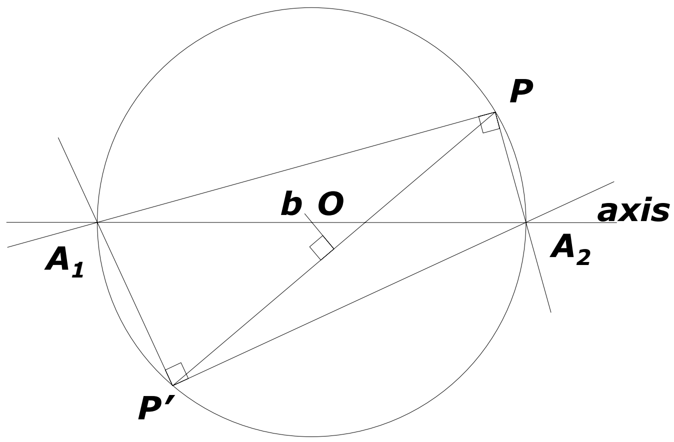
Let the ellipse be given by a pair of conjugate axes. Construct the major and minor axes.



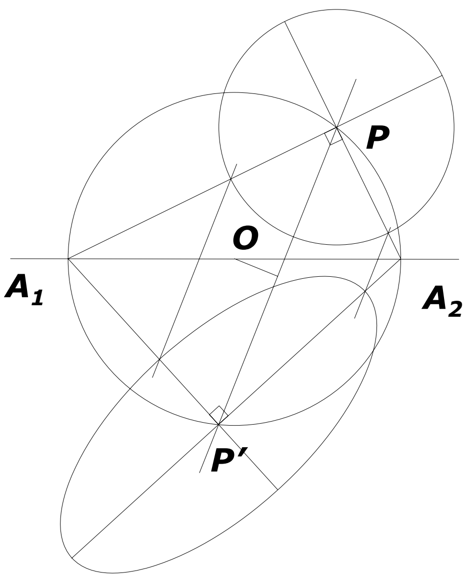


The construction can be derived from the *concentric circles method*. Rotate ***Q*** by -90°. Draw and extend the line segment ***Q*P***. Find the midpoint ***M*** of the segment ***Q*P***. Draw semicircle about ***M*** with the radius ***MO***, find the points of intersection with |***Q*P***|, ***L*** and ***N***. The lines of axes are ***LO*** and ***NO***, the half length of axes are ***a=LP*** and ***b=PN***.

Constructions on Ellipse: Invariant Pair of Right Angles in Oblique Axial Affinity



The perpendicular bisector ***b*** of ***PP′*** intersects the axis at ***O***. The circle about ***O*** through ***P*** and ***P′*** intersects the axis at ***A_1*** and ***A_2***. The affinity transformation preserves the right angle at ***P***.



The axes of an ellipse can be found by means of invariant pair of right angles construction.

Exercises, Affinity

1. Let the affinity be given by ABC → A'B'C' triangles. Find the image of an arbitrary point P and line l.
2. The AB side of the hexagon ABCDEF lies on the axis of affinity of {a, P → P'}. Transform ABCDEF into A'B'C'D'E'F'.
3. Transform a square ABCD with an affinity {a, P → P'}.
4. The a, b and a', b' pairs of intersecting lines determine an axial affinity. Find the image of an arbitrary point P and line l.
5. The axis of affinity and the parallelogram ABCD are given. Determine the axial affinity that transforms ABCD into a square A'B'C'D'.
6. Let the axial (non-orthogonal) affinity be determined by the axis a and a pair of points P → P'. Construct the pair of lines r, s passing through P and r', s' through P' such that both angles of r, s at P and r', s' at P' are right angles.

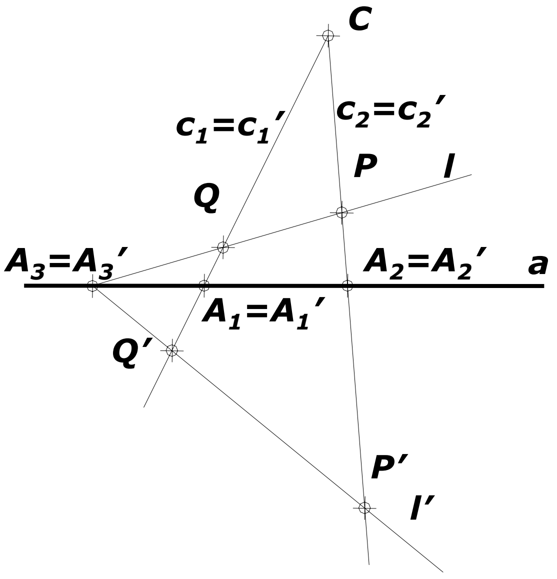
Central-axial Collineation

A *linear mapping* ***Π → Π′*** of the plane onto itself is called *central-axial collineation* with center ***C*** and axis ***a***, if it leaves invariant ***C*** and ***a***. That also means, the lines passing through the center and the points lying in the axis are invariant.

The central-axial collineation is defined by means of the center, axis and a pair of points: {***C, a, P → P′***}***.***

The statement can be proved by showing that for an arbitrary point Q the Q’, for an arbitrary line l the line l’ can be found.

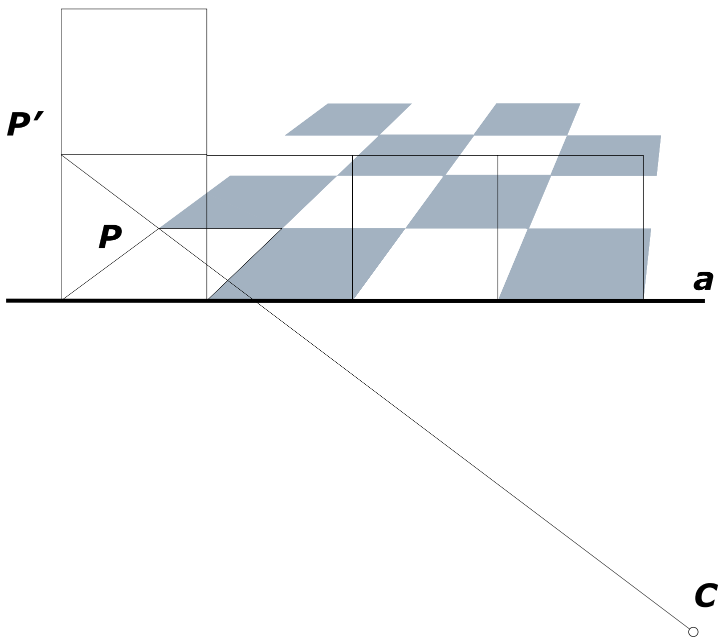
The reverse mapping {***C, a, P′ → P***} ***is also a central-axial collineation.***



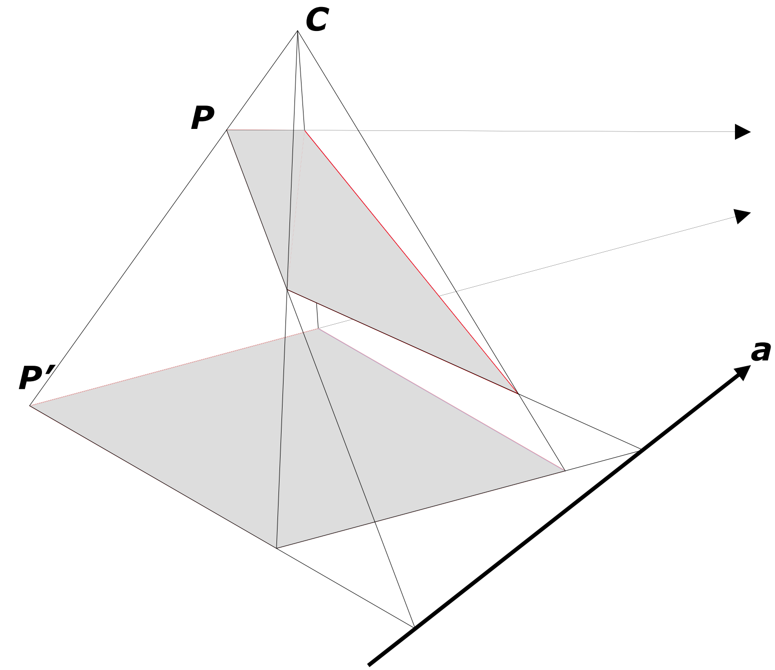
The *central-axial collineation* is also called *perspective collineation.*

Figures in Collineation

Lines parallel to the axis at a perspective collineation remain parallel.

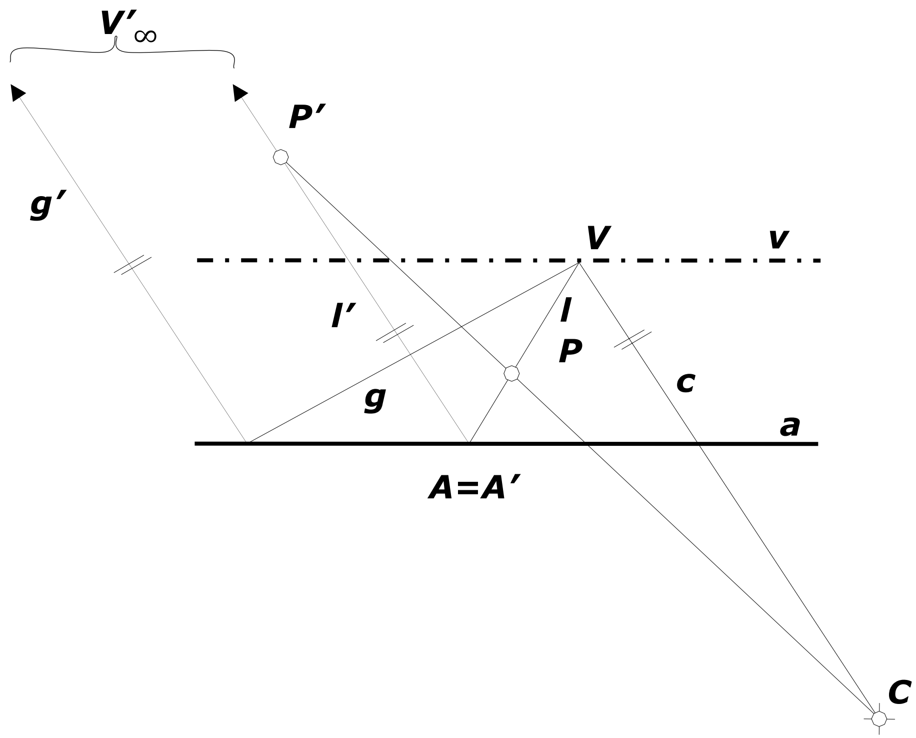


Perspective collineation will be applied at the construction of intersection of pyramid and plane.



Vanishing Line

Point ***V*** is the point of intersection of ***l*** and the line ***c*** parallel to ***l'*** through ***C***. The image of the point ***V*** will be V'_∞ , a "*point at infinity*". If ***g'*** is parallel to ***l***' than ***g*** and ***l*** have the same *vanishing point* ***V***.



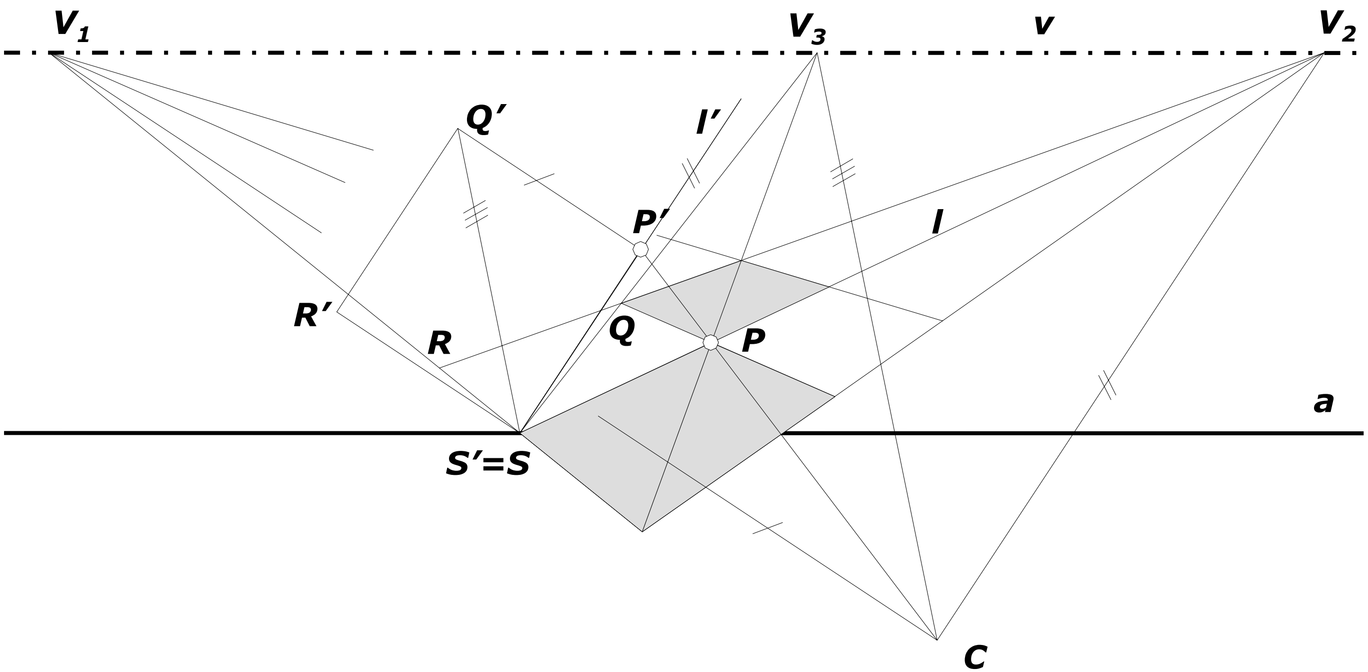
The set of vanishing points is a line ***v*** passing through ***V***, parallel to the axis ***a***.

The mapping $\Pi \rightarrow \Pi'$ of the plane onto itself is complete if it is extended with the image of the line ***v***, an imaginary line i. e. the "*line at infinity*" v'_∞ .

The line v'_∞ is also called "*ideal line*".

Construction by Means of the Vanishing Line

A perspective collineation is determined by the center ***C***, axis ***a*** and the vanishing line ***v***. Let the square ***P′, Q′, R′, S′*** be given. find the quadrilateral ***PQRS*** at the mapping ***Π′ → Π***. (Hint: use an auxiliary line ***l′*** passing through ***P′*** and its image ***l*** that contains the point ***P***. The line ***l*** is determined by ***S*** and the vanishing point V_2)



Chapter Review

Vocabulary

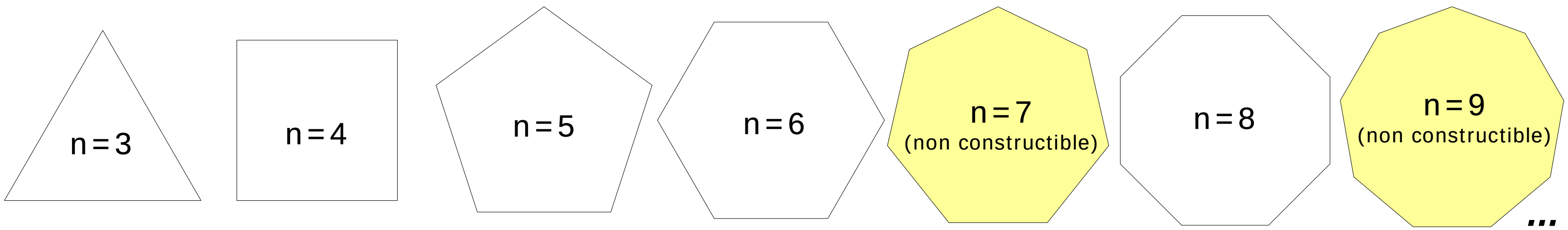
Geometrical transformation
isometry
similarity
scaling
homothecy
dilatation
magnification
shrinking
affinity
axial affinity
collineation
central-axial collineation
invariant elements

Exercises, Collineation

1. Let a central-axial collineation be determined by the axis, center and a pair of corresponding points (collinear with the center). Set up the transformation for the point X’ of any point X, line l’ for any line l.
2. Construct the vanishing line of the central-axial collineation given by the axis, center and a pair of corresponding points (collinear with the center).
3. Let a central-axial collineation be determined by the axis, center and vanishing line. Set up the transformation for the point X’ of any point X, line l’ for any line l.
4. Let a central-axial collineation be determined by the axis, center and a pair of corresponding points (collinear with the center). Construct the reverse image of a square A’B’C’D’.
5. Let a central-axial collineation be determined by the axis, center and vanishing line. Construct the reverse image of a triangle A’B’C’.

Regular Polygons

A polygon is a many-sided shape. A regular polygon is one in which all of the sides and angles are equal. Some examples are shown below.



Only certain regular polygons are "constructible" using the classical Greek tools of the compass and staightedge. According to Gauss´ theorem, a regular *n*-gon can be constructed, if and only if the odd prime factors of *n* are distinct “Fermat primes”

$$F_k = 2^{2^k} + 1.$$

$$F_0 = 2^{2^0} + 1 = 3,\ F_1 = 2^{2^1} + 1 = 5,\ F_2 = 2^{2^2} + 1 = 17, F_3 = 2^{2^3} + 1 = 257, F_4 = 2^{2^4} + 1 = 65537,$$

and it is known, that F_k is composit for $5 \le k \le 32$.

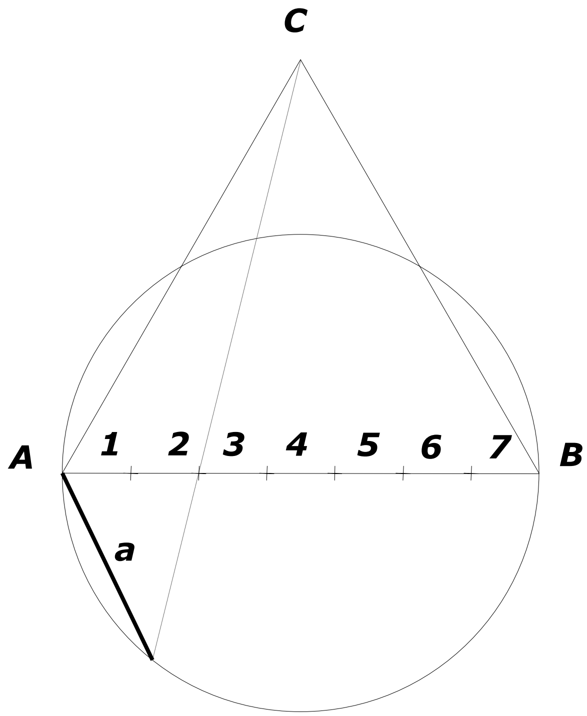
http://mathworld.wolfram.com/FermatNumber.html

http://en.wikipedia.org/wiki/Tilings_of_regular_polygons

http://mathforum.org/dr.math/faq/formulas/faq.regpoly.html

Approximate Construction of Regular Polygons

Approximate construction of regular heptagon:



1) *Draw a diameter **AB** of the circumscribed circle*
2) *Construct an equilateral triangle with the base of the diameter*
3) *Divide the diameter into n=7 equal parts*
4) *Project the second point of division from the vertex **C** of the triangle onto the circle*
5) *segment **a** is the approximate length of the inscribed regular polygon (heptagon)*

About the accuracy of the approximate construction, if the radius of the circle is 10 cm,

a_n is the length of a side of the inscribed polygon, calculated in analytical geometry,

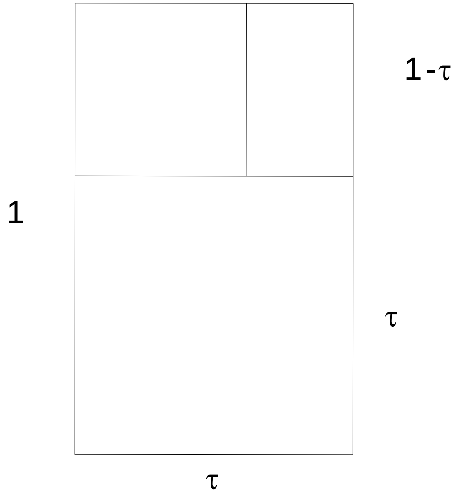
t_n is the length of a side calculated by trigonometry (the "exact" value).

n	3	4	5	6	7	8	9	10	11	12	13
a_n	$10\sqrt{3}$	$10\sqrt{2}$	11.75	10.00	8.69	7.68	6.89	6.23	5.70	5.26	4.87
t_n	$10\sqrt{3}$	$10\sqrt{2}$	11.76	10.00	8.67	7.65	6.84	6.18	5.64	5.18	4.79

Golden Ratio

Divide a segment in such a way that the ratio of the larger part to the smaller is equal to the ratio of the whole to the larger part.

Golden rectangle

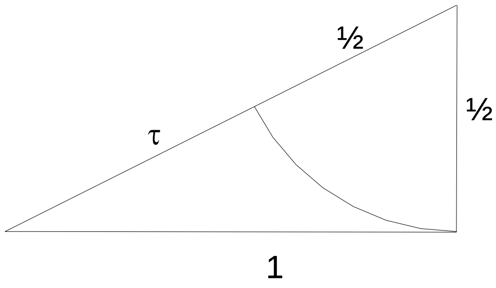


$$\frac{1}{\tau} = \frac{\tau}{1-\tau}$$

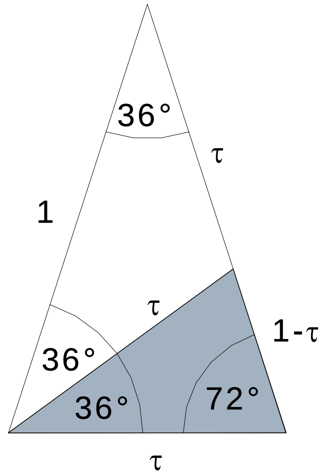
$$\tau^2 + \tau - 1 = 0$$

$$\tau = \frac{\sqrt{5}-1}{2}$$

Construction of τ

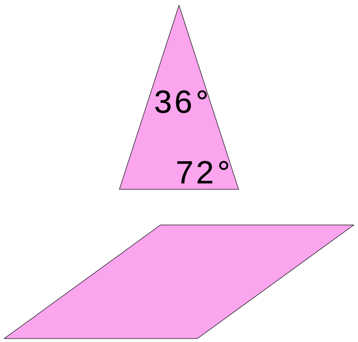


Golden triangle

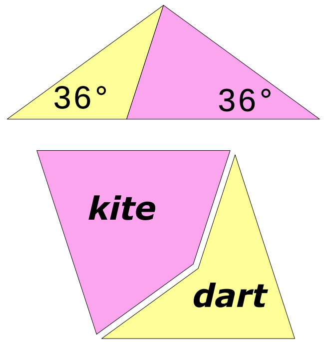


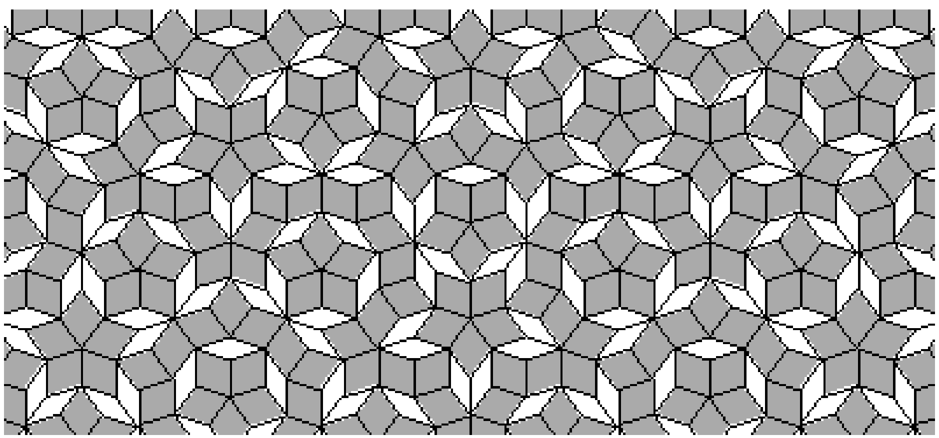
Golden Kite an Dart

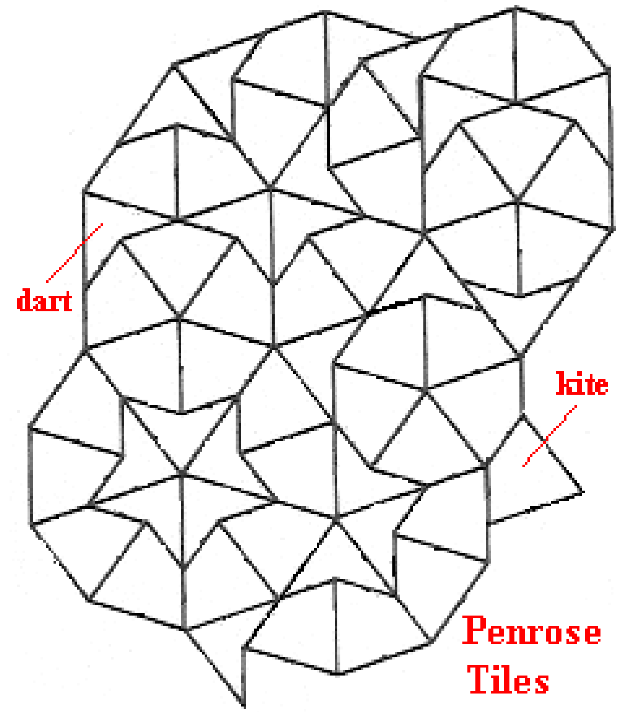
"Sharp" triangle



"Flat" triangle

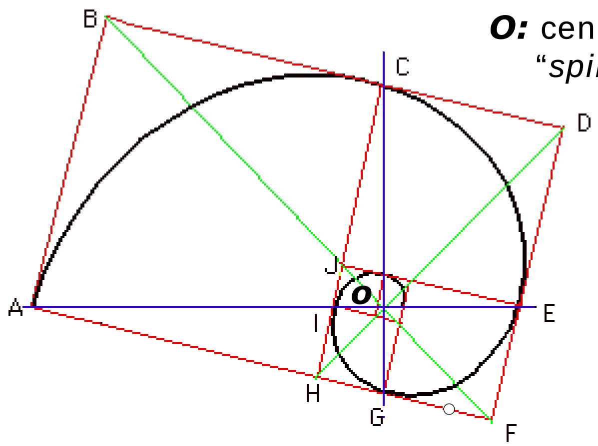




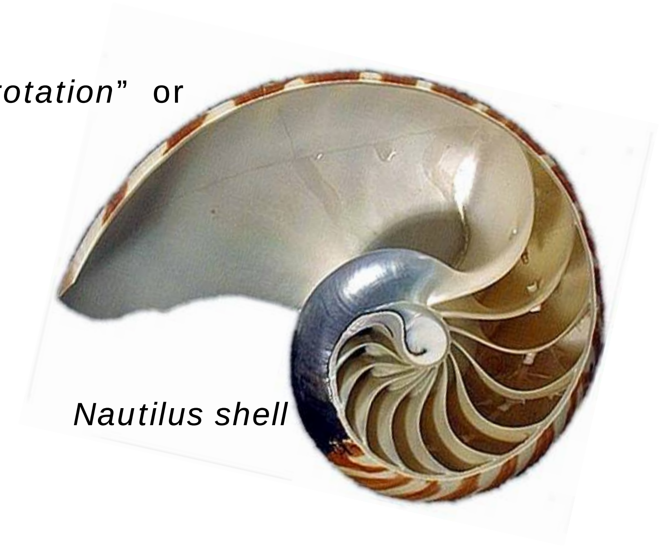


http://goldennumber.net/penrose.htm

Golden Spiral

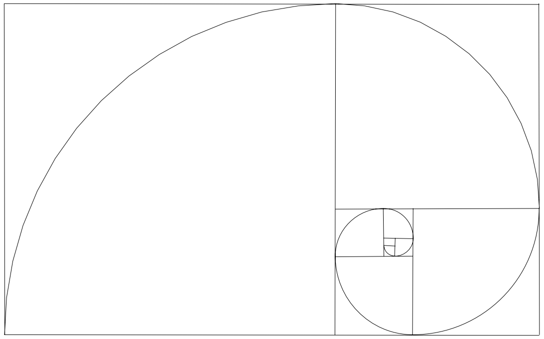


O: center of "*dilative rotation*" or "*spiral similarity*"



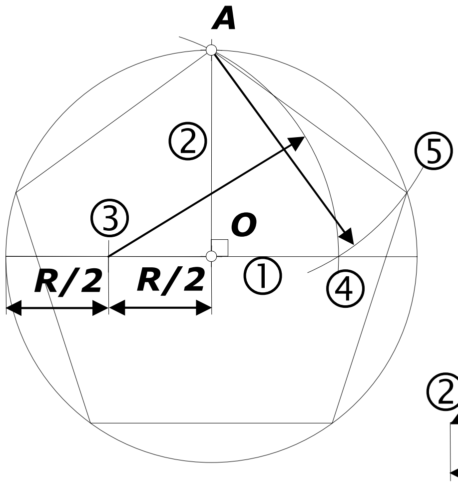
Nautilus shell

The true spiral is closely approximated by the artificial spiral formed by circular quadrants inscribed in the successive squares.

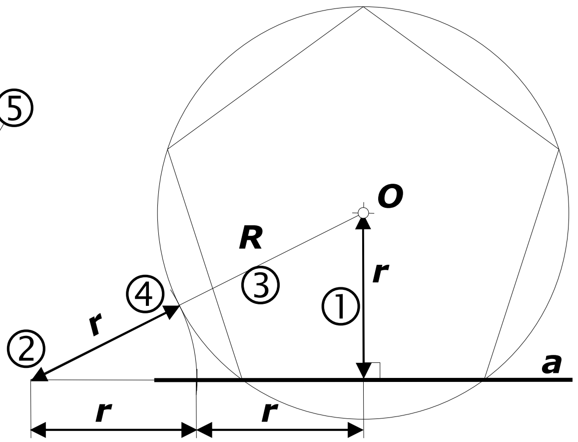


Constructions on Regular Pentagon

Given: ***O, A***



Given: ***O, a***



Given: ***A,B***