Descriptive Geometry 1

by Pál Ledneczki Ph.D.

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About the purposes of studying Descriptive Geometry:

1. **Methods** and "means" for solving 3D geometrical construction problems. In this sense, *Descriptive Geometry* is a branch of Geometry.

2. 2D representation of 3D technical object, i.e. basics of Technical Drawing, “instrument” in technical communication.

What is Descriptive Geometry?

„One simply takes two planes at right angles to each other, one vertical and the other horizontal then projects the figure to be represented orthogonally on these planes, the projections of all edges and vertices being clearly indicated. The projection on the vertical plane is known as the „elevation”, the other projection is called „the plan”. Finally, the vertical plane is folded about the line of intersection of the two planes until it also is horizontal. This puts on one flat sheet of paper what we ordinarily visualize in 3D”.


**Gaspard Monge** (1746 — 1818) was sworn not to divulge the above method and for 15 years, it was a jealously guarded military secret. Only in 1794, he was allowed to teach it in public at the Ecole Normale, Paris where Lagrange was among the auditors. „With his application of analysis to geometry, this devil of a man will make himself immortal”, exclaimed Lagrange.

R.Parthasarathy

About Descriptive Geometry 1

Methodology

- Multi-view representation, auxiliary projections
- Axonomety
- Perspective

Types of problems

- Incidence and intersection problems, shadow constructions
- Metrical constructions
- Representation of spatial elements, polyhedrons, circle
In Descriptive Geometry 1

We shall study

representation of spatial elements and analyze their mutual positions
determine their angles and distances
represent pyramids, prisms, regular polyhedrons,
construct the intersection of polyhedrons with line and plane, intersection of
two polyhedrons
construct shadows
cast shadow, self-shadow, projected shadow

the principles of representation and solution of 3D geometrical
problems in 2D
Spatial elements, relations, notation

Relations

- **pair of points**: determine a distance
- **point and line**: lying on
  - not lying on → plane, distance
- **pair of lines**: coplanar
  - intersecting → angles
  - parallel → distance
  - non coplanar
    - skew → angle and distance
- **point and plane**: lying on
  - not lying on → distance
- **line and plane**: parallel → distance
  - intersecting → angle
- **line and plane**: parallel → distance
  - intersecting → angle
In which quadrant or image plane is the point located, why is it special?

- Point $P$: Located in the $1^{st}$ quadrant.
- Point $P'$: Located in the $2^{nd}$ quadrant.
- Point $P''$: Located in the $3^{rd}$ quadrant.
- Point $P'''$: Located in the $4^{th}$ quadrant.

Multi-view representation
Auxiliary projections

Side view, third image

Chain of transformations

Fourth image, linked to the first image

Multi-view representation
Representation of Straight Lines, Relative Positions

- **First principal line**
- **Second principal line**
- **Profile line**
- **First proj. line**
- **Second proj. line**
- **First coinciding points**
- **Second coinciding points**

**Intersecting**

- \( a'' \)
- \( b'' \)
- \( c'' \)
- \( d'' \)
- \( e'' \)
- \( g'' \)
- \( l'' \)
- \( k'' \)
- \( m'' = n'' \)

**Parallel**

- \( a' \)
- \( b' \)
- \( c' \)
- \( d' \)
- \( e' \)
- \( g' \)
- \( k' = l' \)
- \( m' \)
- \( n' \)

**Skew**

- \( v'' \)
- \( x_{1,2} \)
- \( v' \)
- \( x_{1,2} \)

- \( A'B'' = AB \)
- \( A''B'' \)
- \( C'' \)
- \( D'' \)
- \( v = v' \)
- \( p_1 = K = L' \)

**Multi-view representation**

Descriptive Geometry 1
Point and Line

Descriptive Geometry 1  
Multi-view representation
Tracing Points of a Line

Problems:
1) find the tracing points of principal/profile lines
2) determine lines by means of tracing points
Representation of Plane

Pair of intersecting lines

\[ a'' \quad b'' \]
\[ l'' \quad k'' \]
\[ a' \quad b' \quad k' = l' \]

Pair of parallel lines

\[ c'' \quad d'' \]
\[ m'' = n'' \]
\[ c' \quad d' \]
\[ m' \quad n' \]

Plane figures

\[ A'' \]
\[ B'' \]
\[ C'' \]
\[ 1'' \quad 2'' \quad 3'' \quad 4'' \]

Tracing lines

\[ n_2 \]
\[ n_1 \]
\[ x_{1,2} \]

spanned plane

slanted plane

spanned plane

slanted plane

multi-view representation
Descriptive Geometry 1

Line and Plane

- Lying on (incident) \( l \sim [1234] \)
- Parallel \( \overrightarrow{a} \parallel [ab] \)
- Intersecting \( g \cap [1234] = P \)

Multi-view representation
Intersection of Two Planes

$P_1 = |12| \cap [ABC]$  

$P_2 = |AC| \cap [1234]$  

Multi-view representation
Transversal of a Pair Of Skew Lines Passing Through a Given Point

Sketch and algorithm

Solution in Monge’s

Your solution

\[ B = [Pa] \cap b \]

\[ t = |PB| \]

or

\[ t = [Pa] \cap [Pb] \]

\[ a \parallel a ', [Pa] = [aa] \] Multi-view representation

Descriptive Geometry 1
Transversal of a Pair Of Skew Lines Parallel to a Given Direction

Sketch and algorithm

Solution in Monge’s

Your solution

\[ \begin{align*}
X & \sim a \\
\overrightarrow{d} & \sim X, \ d^* \parallel d \\
B & = \ b \cap [ad^*] \\
t & \sim B, \ t \parallel d^*
\end{align*} \]
Auxiliary Projections on Special Purposes 1

True length of a segment

Distance of a pair of skew lines

Multi-view representation
Auxiliary Projections on Special Purposes 2

Edge view of a plane: transformation of a plane in projecting plane

$$\pi_4 \perp h, \quad \pi_4 \perp \pi_1$$

Application: find the distance $d$ of the point $P$ and the plane $[ABC]$. 

Multi-view representation
Construction of the true shape of a figure lying in a general plane

General plane → fourth projecting plane → fifth principal plane
Cast Shadow, Self-shadow, Projected Shadow
Shadow in Traditional Descriptive Geometry

Riess, C.: Grundzüge der darstellenden Geometrie
(Stuttgart: Verl. J. B. Metzleráschen Buchhandlung, 1871)

Application of Descriptive Geometry for Construction of Projected Shadow (plate X.)

Romsauer Lajos: Ábrázoló geometria (Budapest: Franklin-Társulat, 1929)

http://www.c3.hu/perspektiva/adatbazis/
Shadow in Visualisation

Descriptive Geometry 1

Shadow constructions
Shadows - Basics

Descriptive Geometry 1

Shadow constructions
Shadow Properties

1) Our constructions are restricted to parallel lighting.

2) We do not represent transition between dark and light shade.

3) We usually construct three types of shadow: cast shadow on the ground or on the image planes, self-shadow (shade) and projected shadow.

4) Shadow of a point: piercing point of the ray of light passing through the point, in the surface (on ground plane, picture plane etc.)

5) Shadow of a straight line: intersection of the plane passing through the line, parallel to the direction of lighting and the surface (screen).

6) Shadow of a curve: the intersection of cylinder (whose generatrix is the curve, the generators are rays of light) with the surface (screen).

7) Shadow-coinciding points: pair of distinct points, whose shadows coincide.

8) Alongside cast shadow the surface is in self-shadow.

9) In case of equal orientation of a triangle and its shadow, the face of triangle is illuminated.

10) The cast shadow outline is the shadow of the self-shadow outline.
Cast Shadow, Projected Shadow

Shadow constructions
Intersection of Pyramid and Line

Find the intersection of line and pyramid

auxiliary intersection parallel and similar to the base

auxiliary intersection passing through the apex
Intersection of Polyhedron and Projecting Plane

Find the intersection of plane and polyhedron

Descriptive Geometry 1

Intersection problems
Hint: introduce \( \Pi_4 \) image plane perpendicular to \( \Pi_1 \) and the plane of parallelogram \( (\Pi_4, h, x_{1,4}, h') \).

Intersection of Polyhedron and Plane (auxiliary projection)

Descriptive Geometry 1

Intersection problems
The relation between the base polygon and the polygon of intersection is **central-axial collineation**.

The axis of collineation is the line of intersection of the base plane and the plane of intersection, the center is the apex of the pyramid.

A pair of corresponding points is the pedal point of a lateral edge and the piercing point of the edge in the plane of intersection.

Hint: start with $|MA| \cap [n_1 n_2]$
Intersection of Prism and Line

Find the intersection of line and prism

Auxiliary intersection parallel and congruent to the base

Auxiliary intersection parallel to the generators
The relation between the base polygon and the polygon of intersection is **axial affinity**.

The axis of affinity is the line of intersection of the base plane and the plane of intersection.

A pair of corresponding points is the pedal point of a lateral edge and the piercing point of the edge in the plane of intersection.

**Hint:** start with \(|F'F| \cap [n_1 n_2]|
Intersection of a Pair of Solids

The intersection of two polyhedrons is a polygon (usually 3D polygon).

The vertices of the polygon of intersection are the piercing points of the edges of a polyhedron in the faces of the other polyhedron.

The edges of the polygon of intersection are segments of intersection of pairs of faces.

Sequence: I-VII-III-VIII-V-X-VI-IV-IX-I

At the visibility, one can think of solids or surfaces.

The visibility depends on, what we want to represent as a result of set operation: union, intersection or a kind of difference.
Intersection of a Pair of Solids (your solution)

Algorithm:
Introduce auxiliary image plane perpendicular to the horizontal edges of the prism
Construct the fourth image
Find the piercing points of the edges of pyramid in the faces of the prism
Find the piercing points of the edges of prism in the faces of the pyramid
Find the right sequence of the vertices of polygon of intersection
Draw the polygon of intersection in both images
Show the visibility
Basic Metrical Constructions 1

The true length of a segment is the hypotenuse of right triangle. One of the legs is the length of an image of the segment, the other leg is the difference of distances from the image plane.

Reverse problem: the images of a line, a point of the line and a distance is given. Find the images of points of the line whose true distance from the given point is equal to the given distance. (Hint: by using an auxiliary point of the line find the ratio of the true length and the length of image.)
Any plan geometrical construction can be carried out by rotating the plane parallel to an image plane. The relation between the image of a plane and the image of the rotated plane is *orthogonal axial affinity*. The axis is a principal line of the plane. One rotated point can be found by the true distance of the point and the axis.

Reverse problem: construct the images of a figure, whose rotated image is given. Hint: use inverse affinity and lying on condition.
The first image of a normal of plane, $n'$ is perpendicular to the first image of the first principal line $h'$ of the plane.

The second image of a normal of plane, $n''$ is perpendicular to the second image of the second principal line $v''$ of the plane.

Reverse problem: construct a plane perpendicular to a given line.
Hint: the plane can be determined by means of principal lines.
Modeling of 3D Polyhedrons

Construct a cube. One of the faces is given by its center and line of an edge.

**Algorithm**

1) Construct the square lying in plane \([O,e]\), with the centre \(O\) and an edge on \(e\). (Rotation - counter-rotation of plane, affinity, inverse affinity, (2).)

2) Construct lines perpendicular to the plane \([O,e]\), passing through the vertices of the square. (Perpendicularity of line and plane, (3).)

3) Measure the length of an edge onto the perpendiculare, chose the proper direction from the two possibilities. (True length of a segment, (1).)

4) Complete the figure by showing the visibility.
Step-by-step Construction

Construction of the square  Normal of the plane  Measure of distance  Visibility

Descriptive Geometry 1

Metrical problems
Axonometry

One of the methods of Descriptive Geometry, used to produce pictorial sketches for visualization.

Let three axes $x$, $y$, $z$ through the origin $O$ be given in the image plane. Measure the coordinates from $O$ onto the three axes such that each coordinate will be multiplied by the ratios of foreshortenings $q_x$, $q_y$, $q_z$. The point determined by the coordinates is considered as the axonometric image of the point $P(x, y, z)$.

The axonometric system can be determined by the points $\{O, U_x, U_y, U_z\}$, the image of the origin and the units on the axes $x$, $y$ and $z$.

According to the Fundamental Theorem of Axonometry the axonometric image of an object is a parallel projection or similar to the parallel projection of the object.

In axonometry the left-handed Cartesian system is used.
Let the system \( \{ \mathbf{O}, \mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z \} \) be given in the image plane. The figure \( \{ \mathbf{O}, \mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z \} \) can be considered as the parallel projection of a spatial triplet \( \{ \mathbf{O}, \mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z \} \) of three unit segments perpendicular by pairs. The spatial triplet can be one of two types of orientation apart from translation in the direction of projection.

http://mathworld.wolfram.com/PohlkesTheorem.html
http://www.math-inf.uni-greifswald.de/mathematik+kunst/kuenstler_pohlke.html
Orthogonal Axonometry

\( O' \) is the orthogonal projection of the origin, 
\( X, Y \) and \( Z \) are the piercing points of the axes in the image plane. 
\( O' \) is the orthocenter of the triangle (tracing triangle) \( XYZ \).
Orthogonal Axonometry $\rightarrow$ Multi-view

The orthogonal axonometric image can be considered as one of an ordered pair of images in multi-view representation.
Oblique (klinogonal) Axonometry

\( OO' \) is not perpendicular to the picture plane.
Izometry, Technical Axonometry

Izometry

\[ q_x = q_y = q_z = 1 \]

Technical axonometry

\[ q_x = q_z = 1, \quad q_y = \frac{1}{2} \]
Frontal Axonometry, Shadow
Cavalier, Bird’s-eye View, Worm’s-eye View

Frontal Axonometry

Bird’s eye view (top view)
Military axonometry

Worm’s eye view (bottom view)

Image plane: $[x y]$

if $q_x = q_y = 1$: military axonometry

if $q_z = 1$: cavalier axonometry

Image plane: $[x z]$ if $q_y = 1$:

Image plane: $[y z]$ if $q_x = q_y = 1$
Military Axonometry

Cast Shadow in Orthogonal Axonometry

Dodecahedron, top view and heights
Projected Shadow
Axonometry vs. Perspective
A set of parallel lines in the scene is projected onto a set of lines in the image that meet in a common point. This point of intersection is called the **vanishing point**. A vanishing point can be a finite (real) point or an infinite (ideal) point on the image plane. Vanishing points which lie on the same plane in the scene define a line in the image, the so-called the **vanishing line**.
Basics of Perspective with Vertical Image Plane

Principal point

image plane

ground plane

plane of horizon

image plane

ground plane

plane of horizon

rotated ground plane

rotated center

Perspective
A perspective collineation is determined by the center $C$, axis $a$ and the vanishing line $v$.

To the square $P', Q', R', S' = S$, we can find the quadrilateral $PQRS$ at the mapping $\Pi' \Rightarrow \Pi$.

When the ground plane is rotated into the picture plane, the two systems of points and lines are related by central-axial collineation. This perspective collineation is determined by the center $(C)$, axis $a$ and the horizon $h$.

To the square $(P), (Q), (R), (S) = S$, we can find the quadrilateral $PQRS$ at the mapping $(\Pi) \Rightarrow \Pi$. 
Heights in Perspective

True height can be measured in the image plane.
Shadow in Perspective

http://www.math.ubc.ca/~cass/courses/m309-03a/m309-projects/endersby/Antisolarpoint.html

horizon

If

If

light vanishing point

shadow vanishing point

Descriptive Geometry 1

Perspective
Shadow Types in Perspective
Perspective with Slanting Picture Plane

- Picture plane
- Horizon
- Plane of horizon
- Ground plane
- $V_z$
- $C$
- $F$
- $H$
- $H$
- $F$
- $h$
- $[C]$
- $V_z$
Constructions with Slanting Picture Plane

Rotation of the ground plane

Rotation of a vertical line
Perspective with Slanting Picture Plane
Representation of Circle (Multi-view)

1. The major axes lie on first and second principal lines $h'$ and $v''$ respectively.
2. The length of major axes $1'2'$ and $5''6''$ is equal to the diameter of circle (true length).
3. The length of a minor axis is constructible from the major axis and a point, as plane geometric construction. (See construction of $8''$)
4. The left and right extreme points $9$ and $10$ can be found as points of ellipse with vertical tangents, by means of axial affinity.
5. The tangents at the points mentioned above are parallel to the proper diameters.
1. The major axes of ellipses are perpendicular to the coordinate axes.
2. The minor axes are coinciding lines with the coordinate axes.
3. The fundamental method of constructions is the orthogonal axial affinity.
Representation of Circle (Oblique Axonometry)

1. The fundamental method of constructions is the oblique axial affinity.
2. The axes can be constructed by Rytz’ method. The ellipse is determined by a pair of conjugated diameters; find the major and minor axes.
Representation of Circle (Perspective)

In the ground plane

\[ a \]

**A=(A)**

**O**

**K**

**B**

**h**

**K:** midpoint of AB, center of the ellipse

**O:** image of center of circle

**d** = dist( \(v\), \(a\)) = dist( \(C\), \(h\))

\[ \text{(v)} \] vanishing line of ground plane

\[ \text{dist}((Q),(P)) = \text{dist}((A),(B)) \]

\[ M: \text{midpoint of PQ} \]

In vertical plane

\[ b \]

**M**

**P**

**Q=(Q)**

**V**

**F**

**d**

Distances and relationships are key to understanding the representation of a circle in perspective views.